

THE INTERSECTION DENSITY OF CUBIC ARC-TRANSITIVE GRAPHS WITH 2-ARC-REGULAR FULL AUTOMORPHISM GROUP EQUAL TO $\mathrm{PGL}_2(q)$

Karen Meagher^{*1} and Andriaherimanana Sarobidy Razafimahatratra^{†2}

¹Department of Mathematics and Statistics, University of Regina, Regina, SK, S4S 0A2, Canada
karen.meagher@uregina.ca

²Fields Institute for Research in Mathematical Sciences, 222 College St, Toronto, Canada
sarobidy@phystech.edu

Submitted: Mar 22, 2025; Accepted: Aug 14, 2025; Published: Jan 15, 2026

© The authors. Released under the CC BY license (International 4.0).

Abstract. The *intersection density* of a transitive permutation group $G \leq \mathrm{Sym}(V)$ is the ratio between the largest size of a subset of G in which any two agree on at least one element of V , and the order of a point-stabilizer of G . In this paper, we determine the intersection densities of the automorphism groups of the arc-transitive graphs admitting a 2-arc-regular full automorphism group $G^* = \mathrm{PGL}_2(q)$ and an arc-regular subgroup of automorphism $G = \mathrm{PSL}_2(q)$.

Keywords. Derangement graphs, cocliques, projective special linear groups

Mathematics Subject Classifications. 05C35, 05C69, 20B05

1. Introduction

Given a transitive permutation group $G \leq \mathrm{Sym}(V)$ where V is a finite non-empty set, a subset $\mathcal{F} \subset G$ is *intersecting* if for any $f, g \in \mathcal{F}$, there exist $v \in V$ such that $f(v) = g(v)$. If $\mathcal{F} \subset G$ is intersecting and $g \in \mathcal{F}$, then the set $g^{-1}\mathcal{F} = \{g^{-1}x : x \in \mathcal{F}\}$ is an intersecting set containing the identity of G . Therefore, we may assume without loss of generality that any intersecting set contains the identity of G . The *intersection density* of $G \leq \mathrm{Sym}(V)$ is the rational number

$$\rho(G) = \max \frac{|\mathcal{F}| : \mathcal{F} \subset G \text{ is intersecting}}{|G|/|V|}.$$

*Research supported in part by an NSERC Discovery Research Grant, Application No.: RGPIN-2025-04101.

†The author gratefully acknowledges that this research was supported by the Fields Institute for Research in Mathematical Sciences.

If G_v is the stabilizer of $v \in V$ in G , then G_v is intersecting of size $\frac{|G|}{|V|}$, and so $\rho(G) \geq 1$.

The notion of intersection density was first introduced in [LSP20] to measure how large the intersecting sets in a given transitive group can be compared to its point stabilizers. The majority of the work prior to [LSP20] focused on transitive groups with intersection density equal to 1, see [FD77, GM09, MR21, MS11, MST16] for example. In the past few years, the study of intersection densities of transitive groups has become a very active research area. For instance, it was shown in [MRS21] that the intersection density of a transitive group is at most a third of the degree of the group, and it was recently shown in [CGS25] that there are only four transitive groups with intersection densities attaining this upper bound. In [BMR24, HKKM25, HKMM22, LSP20], the intersection densities of transitive groups with prescribed degrees were studied. The notion of intersection density was also recently extended to vertex-transitive graphs in [KMP23, MR24].

In this paper, we study the intersection densities of certain automorphism groups of cubic arc-transitive graphs. Recall that a graph $X = (V, E)$ is cubic if every vertex has degree 3. An *arc* in a graph is an ordered pair of adjacent vertices, and a graph is *arc-transitive* if its automorphism group acts transitively on the set of arcs. In general, an *s-arc* of X is an $(s+1)$ -tuple $(u_0, u_1, u_2, \dots, u_s)$ with vertices such that $\{u_i, u_{i+1}\} \in E$ for $0 \leq i \leq s-1$, with the condition that $u_{i-1} \neq u_{i+1}$ for $1 \leq i \leq s-1$. The graph X is called *s-arc-transitive* if its automorphism group acts transitively on s -arcs, and *s-arc-regular* if it acts regularly on s -arcs.

Given a vertex-transitive graph $X = (V, E)$, the *intersection density array* of X is the increasing array

$$\rho(X) := [\rho_1, \rho_2, \dots, \rho_t] \quad (1.1)$$

such that for any transitive subgroup $H \leq \text{Aut}(X)$, there exists $j \in \{1, 2, \dots, t\}$ such that $\rho(H) = \rho_j$, and for any $j \in \{1, 2, \dots, t\}$, there is at least one transitive $H \leq \text{Aut}(X)$ whose intersection density is ρ_j .

For example, the intersection density array of the Petersen graph is $[1, 2]$ since its full automorphism group is $\text{Sym}(5)$, and the only other transitive subgroups are $\text{Alt}(5)$ and $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$. The intersection densities of $\text{Sym}(5)$ and $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ are both 1, while $\text{Alt}(5)$ has intersection density 2.

The *weak intersection density array* of an arc-transitive graph X is the sub-array $\bar{\rho}(X)$ consisting of all $\rho \in \rho(X)$ that are realized by arc-transitive subgroups of automorphism.

Cubic arc-transitive graphs have been extensively studied and they have been classified through their automorphism groups. For a comprehensive review of cubic arc-transitive graphs, the reader is referred to [DMM22, Chapter 10]. A cubic arc-transitive graph $X = (V, E)$ is of *type* $\{1, 2^1\}$ if its full automorphism group G^* is 2-arc-regular, admitting a 1-arc-regular subgroup G , and its edge stabilizer in G^* is isomorphic to \mathbb{Z}_2^2 . In [KMP23], Kutnar, Marušič and Pujol initiated the study of the intersection densities of transitive automorphism groups of cubic arc-transitive graphs. In particular, they introduced the notion of weak intersection density array and considered it for this family of graphs. By definition, cubic arc-transitive graphs of type $\{1, 2^1\}$ have a 2-arc-regular full automorphism group, G^* and a 1-arc-regular transitive subgroup $G \leq G^*$. Hence, the weak intersection density array of such graph is $[\rho(G^*), \rho(G)]$ if $\rho(G) > \rho(G^*)$, and $[\rho(G)]$ if $\rho(G) = \rho(G^*)$.

In [KMP23], under some strong restrictions, the weak intersection density array of many of these graph was computed. In this paper, we continue this work by determining the weak intersection density array of the cubic arc-transitive graphs of type $\{1, 2^1\}$ with $G^* = \text{PGL}_2(q)$ and $G = \text{PSL}_2(q)$, where q is an odd prime power. Whenever $q = 2^k$, there are no graphs of type $\{1, 2^1\}$ since $G^* = G = \text{PGL}_2(q)$, so we henceforth assume that p is an odd prime. We will assume the following hypothesis for convenience.

Hypothesis 1. *Let $X = (V, E)$ be an arc-transitive cubic graph of type $\{1, 2^1\}$ with full automorphism group $G^* = \text{PGL}_2(q)$, where $q = p^k$ is an odd prime power, and $G = \text{PSL}_2(q)$ is a 1-arc-regular subgroup of G^* .*

Marston Conder [Con25] gives a complete list of the cubic symmetric graphs on up to 10000 vertices, of these he identifies 1135 which are of type 2^1 . It is possible to check this list for the graphs whose automorphism group satisfies the hypothesis. Doing this shows that 57 of these graphs satisfy our hypothesis, these are listed the appendix.

If X, G and G^* are as in Hypothesis 1, then for $v \in V$, there exists $h \in G$ of order 3 and an involution $r \in G^* \setminus G$ such that $rh r^{-1} = h^{-1}$, $G_v = \langle h \rangle$ and $G_v^* = \langle h, r \rangle \cong D_6 \cong \text{Sym}(3)$. Under the action of $G^* = \text{PGL}_2(q)$ on X , exactly the elements conjugate to an element in $G_v^* \cong \text{Sym}(3)$ have a fixed point; in particular, $g, h \in G^*$ are intersecting exactly when hg^{-1} is conjugate to an element in G_v^* . Similarly, with this action, the elements with a fixed point in G are exactly those conjugate to an element in $\langle h \rangle$.

As $|G| = \frac{(q-1)q(q+1)}{2}$ and $|G^*| = (q-1)q(q+1)$, it is clear that G and G^* always have elements of order 2 and 3. In fact, unless $q = 9$, $G = \text{PSL}_2(q)$ is generated by an involution and an element of order 3. Furthermore, G^* is always an extension of G by a cyclic group of order 2, whenever q is odd.

For q a power of 3, the following result was proved in [HKKM25, Theorem 7.2] and also in [KMP23, Theorem 4.8].

Theorem 1.1. *If $G = \text{PSL}_2(3^k)$ and $G^* = \text{PGL}_2(3^k)$ with $k \geq 3$ an integer, is as in Hypothesis 1, then*

$$\rho(G^*) = \rho(G) = \begin{cases} 3^{k-1} & \text{if } k \text{ is odd} \\ 3^{\frac{k}{2}-1} & \text{otherwise.} \end{cases}$$

The next result was proved in [HKKM25, Theorem 6.1] for the case when $q \equiv 1 \pmod{3}$.

Theorem 1.2. *If $q = p^k \equiv 1 \pmod{3}$, and $G = \text{PSL}_2(q)$ is as in Hypothesis 1, then*

$$\rho(G) = \begin{cases} \frac{4}{3} & \text{if } p \neq 5 \\ 2 & \text{if } p = 5. \end{cases}$$

Following Theorem 1.1 and Theorem 1.2, the only open cases for both $G = \text{PSL}_2(q)$ and $G^* = \text{PGL}_2(q)$ are when $q = p^k \equiv 2 \pmod{3}$; in this paper we resolve these cases. Note if $q = p^k \equiv 2 \pmod{3}$, then k is odd and $p \equiv 2 \pmod{3}$. Theorem 1.2 computes the intersection density of arc-regular subgroups in Hypothesis 1, but not of the full automorphism groups. So for $q \equiv 1 \pmod{3}$, we will also determine the intersection density of $G^* = \text{PGL}_2(q)$.

The main results of this paper, stated together with the known results, is the following.

Theorem 1.3. *If X is a cubic arc-transitive graph as in Hypothesis 1, with $q = p^k$ an odd prime power, and $(G, G^*) = (\text{PSL}_2(q), \text{PGL}_2(q))$, then*

$$\bar{\rho}(X) = \begin{cases} [3^{k-1}] & \text{if } q = 3^k \text{ and } k \text{ odd,} \\ [3^{\frac{k}{2}-1}] & \text{if } q = 3^k \text{ and } k \text{ even,} \\ [1, 2] & \text{if } q \equiv 1 \pmod{3} \text{ and } p = 5, \\ [1, \frac{4}{3}] & \text{if } q \equiv 1 \pmod{3} \text{ and } p \neq 5, \\ [1] & \text{if } q \equiv 2 \pmod{3} \text{ and } q \equiv \pm 2 \pmod{5}, \\ [1, \frac{4}{3}] & \text{if } q \equiv 2 \pmod{3} \text{ and } q \equiv \pm 1 \pmod{5} \text{ or } q \equiv 0 \pmod{5}. \end{cases}$$

2. Background results

2.1. Orbital graphs

Throughout this section, we let $G \leq \text{Sym}(V)$ be a finite transitive group. The group G acts intransitively on $V \times V$, and an orbit for this action is called an **orbital** of G . By transitivity of G , the set $O_0 = \{(v, v) : v \in V\}$ is an orbital, called the trivial orbital, or the diagonal orbital of G . Any other orbital of $G \leq \text{Sym}(V)$ is a subset of $\{(u, v) : u, v \in V, u \neq v\}$.

Let O be a non-trivial orbital of G . The set $O^* = \{(v, u) : (u, v) \in O\}$ is also an orbital of G , and if O is an orbital such that $O = O^*$, then O is called a **self-paired orbital**. The orbital O determines a digraph $X_O = (V, O)$, called an **orbital graph**, whose vertex set is V , and whose arc set is O . If O is a self-paired orbital, then X_O may be viewed as an undirected graph, otherwise, X_O is an oriented digraph. A **basic orbital graph** of G is an undirected graph of the form X_O for some self-paired orbital O , or $X_O \cup X_{O^*}$ for some orbital O of G that is not self-paired.

Recall that a graph $X = (V, E)$ is called **vertex transitive** if $\text{Aut}(X)$ acts transitively on V . The next result shows that any vertex-transitive graph can be reconstructed from orbital digraphs.

Lemma 2.1. *[DMM22, Proposition 1.4.6] Let $X = (V, E)$ be a vertex-transitive graph. If $G \leq \text{Aut}(X)$ is transitive, then X is a union of basic orbital graphs of G .*

A vertex-transitive graph $X = (V, E)$ is called a **Cayley graph** if there exists $H \leq \text{Aut}(X)$ acting regularly on V , that is, for any $u, v \in V$, there exists a unique $h \in H$ such that $v = h(u)$. If X is a Cayley graph, then X is isomorphic to a graph $\text{Cay}(H, C)$, where C is an inverse-closed

subset of $H \setminus \{1\}$, with vertex set H and two elements h and h' in H are adjacent if and only if $h'h^{-1} \in C$.

Next, we recall an important correspondence between orbital graphs and the suborbits of a transitive subgroup of automorphisms. Recall that the **suborbits** of $G \leq \text{Sym}(V)$ are the orbits of G_v , for some $v \in V$. We will assume henceforth that $O_0, O_1, O_2, \dots, O_d$ are the orbitals of $G \leq \text{Sym}(V)$, where O_0 is the trivial orbital. Fix $v \in V$. For any $0 \leq i \leq d$, there exists $w_i \in V$ such that O_i is equal to the orbital of G containing (v, w_i) . By transitivity of G on O_i , it is clear that G_v is also transitive on the set $\Delta_i = \{w \in V : (v, w) \in O_i\}$. Therefore, there is a one-to-one correspondence between the orbitals of G and the orbits of G_v , that is, the suborbits of G with respect to v . In particular, for any $0 \leq i \leq d$ the orbital O_i of G that contains (v, w_i) , corresponds to the suborbit Δ_i containing w_i . We say that a suborbit of G is self-paired if the corresponding orbital is self-paired. Hence, we also define the graph $X_{\Delta_i} := X_{O_i}$ for $1 \leq i \leq d$.

The following result gives a characterization of arc-transitive digraphs.

Lemma 2.2. [DMM22, Theorem 3.2.8] *Let $X = (V, E)$ be a graph, $G \leq \text{Aut}(X)$ be transitive and $v \in V$. The following statements are equivalent.*

1. X is an arc-transitive graph,
2. the neighbourhood $N_X(v)$ of v in X is a self-paired suborbit of G ,
3. X is an orbital graph of G .

2.2. Cubic arc-transitive graphs

Let $X = (V, E)$ be a cubic arc-transitive graph of type $\{1, 2^1\}$ with $G^* = \text{Aut}(X)$. By Lemma 2.2, the graph X is isomorphic to an orbital graph X_Δ of any transitive subgroup of G^* , where Δ is a self-paired suborbit of size 3. Fix $v \in V$ and let $\Delta = N_X(v) = \{u_1, u_2, u_3\}$. Since G is a transitive subgroup of G^* , we know that G_v acts transitively on Δ . Using the fact that G is 1-arc regular, the subgroup G_v acts regularly on the set of arcs $\{(v, u_1), (v, u_2), (v, u_3)\}$. Therefore, G_v must be a cyclic group of order 3. We assume that $G_v = \langle h \rangle$, for some $h \in G$ of order 3. By the orbit-stabilizer lemma, we know that $|V| = \frac{|G|}{|G_v|} = \frac{q(q^2-1)}{2(3)}$. As Δ is also a suborbit of G^* , the orbit-stabilizer lemma implies that $|G_v^*| = \frac{|G^*|}{|V|} = 6$. Since G_v^* is transitive on Δ , there exists an involution $r \in G_v^*$ such that $rhr = h^{-1}$, and $G_v^* = \langle h, r \rangle$.

2.3. Derangement graphs

Let $G \leq \text{Sym}(V)$ be a transitive group. The derangement graph is an important tool to determine the intersection density of transitive groups. The **derangement graph** Γ_G is the graph with vertex set G , and two elements g and h are adjacent if hg^{-1} is a derangement (that is, a fixed-point-free permutation). An important property of the stabilizers of G is that if $u, v \in V$ and $g \in G$ such that $g(u) = v$, then $G_u = g^{-1}G_vg$. Therefore, g and h are adjacent in Γ_G if and only if hg^{-1} is not conjugate to any element in G_v , for any fixed $v \in V$.

If D_G is the set of derangements of G , then it is not hard to see that $\Gamma_G = \text{Cay}(G, D_G)$. The derangement graph Γ_G is defined in a way that \mathcal{F} is intersecting with respect to the action of G on V if and only if \mathcal{F} is a coclique of Γ_G . Therefore, one can extend the definition of the intersection density for G acting on V as follows

$$\rho(G) = \frac{\alpha(\Gamma_G)}{|G|/|V|}.$$

3. Properties of the linear groups $\text{PSL}_2(q)$ and $\text{PGL}_2(q)$

In this section, we determine the properties of $\text{PSL}_2(q)$ and $\text{PGL}_2(q)$ that we will need. This includes the structure of the cyclic subgroups of $\text{PSL}_2(q)$. Along the way, we will give the normalizer and the point-stabilizers of these elements. We start with some remarks about the elements of $\text{PSL}_2(q)$ and $\text{PGL}_2(q)$. We will assume q is an odd prime power.

3.1. Properties of elements

Any element of $\text{PSL}_2(q)$ is conjugate to an element of a certain cyclic subgroup. In particular, if $g \in \text{PSL}_2(q)$ is of order $k > 2$, then g is conjugate to an element of a subgroup isomorphic to: $\mathbb{Z}_{\frac{q-1}{2}}$, if $k \mid \frac{q-1}{2}$; $\mathbb{Z}_{\frac{q+1}{2}}$, if $k \mid \frac{q+1}{2}$; and \mathbb{Z}_p , if $k \mid q$. An involution of $\text{PSL}_2(q)$ is conjugate to an element in $\mathbb{Z}_{\frac{q-1}{2}}$ if $q \equiv 1 \pmod{4}$, and to an element in $\mathbb{Z}_{\frac{q+1}{2}}$ if $q \equiv 3 \pmod{4}$. The normalizers of the cyclic subgroups of $\text{PSL}_2(q)$ are given in the next table; the table for $\text{PGL}_2(q)$ is given following some comments.

Congruence of q	$o(g) = 2$	$o(g) \mid \frac{q-1}{2}$	$o(g) \mid \frac{q+1}{2}$	$o(g) \mid q$
$q \equiv 1 \pmod{4}$	D_{q-1}	D_{q-1}	D_{q+1}	$\mathbb{Z}_p^k : \mathbb{Z}_{\frac{p-1}{2}}$
$q \equiv 3 \pmod{4}$	D_{q+1}	D_{q-1}	D_{q+1}	$\mathbb{Z}_p^k : \mathbb{Z}_{\frac{p-1}{2}}$

Table 3.1: The normalizer $N_G(\langle g \rangle)$ of $\langle g \rangle$ in $G = \text{PSL}_2(q)$, when q is an odd prime power.

Similar to the case for $\text{PSL}_2(q)$, an element $g \in \text{PGL}_2(q)$ of order $k > 2$ is conjugate to an element in a cyclic group isomorphic to: \mathbb{Z}_{q-1} , if $k \mid (q-1)$; \mathbb{Z}_{q+1} , if $k \mid (q+1)$; and \mathbb{Z}_p if $k \mid q$. One main difference between $\text{PSL}_2(q)$ and $\text{PGL}_2(q)$ is the structure of involutions. For $\text{PSL}_2(q)$ there is a unique conjugacy class of involutions, these involutions are contained in either D_{q-1} or D_{q+1} , depending on the congruence of q modulo 4. For $\text{PGL}_2(q)$, there are two conjugacy classes of involutions if q is odd (but only one if q is even). For $\text{PGL}_2(q)$, the normalizer of cyclic subgroups are given below.

3.2. Dihedral subgroups of order 6

As the point-stabilizers of the full automorphism groups that we study in this paper are isomorphic to $D_6 \cong \text{Sym}(3)$, we need to recall some facts about the conjugacy classes of these subgroups in $\text{PGL}_2(q)$.

Congruence of q	$o(g) = 2, g \in \text{PSL}_2(q)$	$o(g) = 2, g \notin \text{PSL}_2(q)$	$o(g) \mid (q - 1)$	$o(g) \mid (q + 1)$	$o(g) = p$
$q \equiv 1 \pmod{4}$	$D_{2(q-1)}$	$D_{2(q+1)}$	$D_{2(q-1)}$	$D_{2(q+1)}$	$\mathbb{Z}_p^k : \mathbb{Z}_{p-1}$
$q \equiv 3 \pmod{4}$	$D_{2(q+1)}$	$D_{2(q-1)}$	$D_{2(q-1)}$	$D_{2(q+1)}$	$\mathbb{Z}_p^k : \mathbb{Z}_{p-1}$

Table 3.2: The normalizer $N_{G^*}(\langle g \rangle)$ of $\langle g \rangle$ in $G^* = \text{PGL}_2(q)$, when q is an odd prime power.

Lemma 3.1. [COTR06] *Let $q = p^k$ be odd. The number of conjugacy classes of subgroups of $\text{PGL}_2(q)$ isomorphic to the dihedral group D_6 are as follows.*

1. *If $p \neq 3$, then there are two conjugacy classes of subgroups isomorphic to D_6 . In particular, one class lies in $\text{PSL}_2(q)$ and the other does not.*
2. *If $q = 3^k$, then there is a unique conjugacy class of subgroups isomorphic to D_6 . If k is even, this class lies in $\text{PSL}_2(q)$, and if k is odd, then it does not.*

If q is odd and not a power of 3, then there are two conjugacy classes of subgroups isomorphic to $D_6 \cong \text{Sym}(3)$, so we let $H \leq \text{PSL}_2(q)$ and $H' \not\leq \text{PSL}_2(q)$ be representatives of these two conjugacy classes. Since neither of H or H' contain $\text{PSL}_2(q)$, H and H' are core-free subgroups. Consequently, the actions of $\text{PGL}_2(q)$ on cosets of H and H' by multiplication are faithful, and thus correspond to transitive permutation groups.

Let V' be the set of cosets of H' in $\text{PGL}_2(q)$, we claim that the action of $\text{PSL}_2(q)$ on V' is transitive. To see this, let $r \in \text{PGL}_2(q) \setminus \text{PSL}_2(q)$ be an involution. Then, $\text{PGL}_2(q) = \text{PSL}_2(q) \rtimes \langle r \rangle$. If xH' and yH' are two cosets in V' , then there exist $x', y' \in \text{PSL}_2(q)$ such that $x = x'r$ and $y = y'r$. Then, we have

$$(y'(x')^{-1})(xH') = (y'(x')^{-1})(x'rH') = y'rH' = yH'.$$

Let V be the set of cosets of H in $\text{PGL}_2(q)$, we claim that the action of $\text{PSL}_2(q)$ on V is intransitive. Fix $v \in V$ and let $g \in \text{PGL}_2(q) \setminus \text{PSL}_2(q)$ such that $u = g(v)$. Then assume that there is a $g' \in \text{PSL}_2(q)$, such that $u = g'(v)$. This implies $g^{-1}g'(v) = v$, and $g^{-1}g'$ is conjugate to an element in H , so conjugate to an element in $\text{PSL}_2(q)$. But since $g^{-1}g' \in \text{PGL}_2(q) \setminus \text{PSL}_2(q)$, this is not possible and no such g' exists in $\text{PSL}_2(q)$.

Consequently, we conclude that if q is not a power of 3 and $\text{PSL}_2(q)$ acts transitively on the cosets of a subgroup of $\text{PGL}_2(q)$ isomorphic to D_6 , then the subgroup is conjugate to H' . In particular, the point stabilizers of this action are conjugate to H' .

Next consider $q = 3^k$ for some integer $k \geq 1$, by Lemma 3.1 in this case there is a unique conjugacy class of subgroups of isomorphic to $\text{Sym}(3)$. If k is even, then a copy of $\text{Sym}(3)$ given by the normalizer of an element of order 3 in $\text{PSL}_2(q)$ lies in $\text{PSL}_2(q)$. Consequently, using the same argument before, $\text{PSL}_2(q)$ cannot be transitive on V in this case. If k is odd, no subgroup of $\text{PGL}_2(q)$ isomorphic to $\text{Sym}(3)$ lies in $\text{PSL}_2(q)$. Similar to the case where q is not a power of 3, the subgroup $\text{PSL}_2(q)$ is transitive.

We summarize what we showed in this section, using the fact that Hypothesis 1 implies the action of $\text{PSL}_2(q)$ is transitive.

Lemma 3.2. *Let X be a cubic arc-transitive graph of type $\{1, 2^1\}$ satisfying Hypothesis 1 with $G^* = \text{PGL}_2(q)$ and $G = \text{PSL}_2(q)$, then no vertex-stabilizer of $\text{PGL}_2(q)$ can lie in $\text{PSL}_2(q)$. In particular, if $q = 3^k$, then k is odd.*

3.3. The elements of order 3 and 2 in $\text{PSL}_2(q)$

The next result, from [HKKM25, Proposition 2.12] describes the elements of order 3 in $\text{PSL}_2(q)$, where $q = p^k$ for some odd prime number p and some integer $k \geq 1$.

Lemma 3.3. [HKKM25, Proposition 2.12] *Let $\bar{A} \in \text{PSL}_2(q)$ be an element such that $\bar{A} \neq \pm 1$. Then \bar{A} has order 3 if and only if $\text{Tr}(A) = \pm 1$.*

Next we determine the conjugacy classes of involutions of $\text{PGL}_2(q)$ not belonging to $\text{PSL}_2(q)$.

Lemma 3.4. *If $\bar{A} \in \text{PGL}_2(q) \setminus \text{PSL}_2(q)$ is an involution, then $\text{Tr}(A) = 0$.*

Proof. We note that $\text{PGL}_2(q)$ has two conjugacy classes of involutions. The first class consists of the involutions in $\text{PSL}_2(q)$ and the second one those in $\text{PGL}_2(q) \setminus \text{PSL}_2(q)$. Every involution in $\text{PGL}_2(q) \setminus \text{PSL}_2(q)$ is conjugate to the element

$$\overline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}.$$

The result follows immediately by noting that the trace of the above element is 0. □

3.4. Transversals

In this section we build a subgroup of order $q+1$ in $\text{PGL}_2(q)$ and determine a transversal for this subgroup. To do this, consider an element h with order 3 in $\text{PGL}_2(q)$, we can assume without loss of generality that

$$h = \overline{\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}}.$$

Clearly, $\text{PGL}_2(q)$ acts transitively on the conjugacy class that includes h , acting by conjugation. The point-stabilizer of this action is $C_{\text{PGL}_2(q)}(h)$. The centralizer of h , as matrix in $\text{GL}_2(q)$, is

the set of matrices $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$

or, equivalently, with

$$\begin{bmatrix} -c & -d \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} b & -a-b \\ d & -c-d \end{bmatrix}.$$

Consequently, $b = -c$ and $d = a + b$. From these equations, we have the centralizer of h in $\text{PGL}_2(q)$ is

$$\bar{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a+b \end{bmatrix} = \begin{cases} \overline{\begin{bmatrix} 1 & \alpha \\ -\alpha & 1+\alpha \end{bmatrix}} & \text{for some } \alpha \in \mathbb{F}_q \text{ if } a \neq 0, \\ \overline{\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}} & a = 0. \end{cases}$$

We therefore conclude that the centralizer of h in $\text{PGL}_2(q)$ is the subgroup

$$S = C_{\text{PGL}_2(q)}(h) = \left\{ \overline{\begin{bmatrix} 1 & \alpha \\ -\alpha & 1+\alpha \end{bmatrix}} : \alpha \in \mathbb{F}_q \right\} \cup \left\{ \overline{\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}} \right\} \cong \mathbb{Z}_{q+1}. \tag{3.1}$$

In Lemma 4.1 we will determine the suborbits of S acting by conjugation on the conjugacy class that contains h . In order to do this, we first need to identify some representatives of cosets of S . Define the subgroup

$$K = \left\{ \overline{\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}} : a \in \mathbb{F}_q \text{ and } b \in \mathbb{F}_q^* \right\} \cong \mathbb{F}_q \rtimes \mathbb{Z}_{q-1}. \tag{3.2}$$

Proposition 3.5. *The subgroup K is a left-transversal of S in $\text{PGL}_2(q)$.*

Proof. Note that $|S| = q + 1$ and $|K| = q(q - 1)$, so the elements in $S \cap K$ cannot have order more than 2. As $2 \mid (q + 1)$ and $2 \mid (q - 1)$, S and K both admit involutions, however, the involutions in S are conjugate to elements of $\text{PSL}_2(q)$, whereas the involutions in K cannot be conjugate to any element of $\text{PSL}_2(q)$. Therefore, $|S \cap K| = 1$. Since $S \cap K$ is trivial and K is a subgroup, if $x, y \in K$ such that $x^{-1}y \in S$, then $x = y$. \square

From this proposition, we deduce that $\text{PSL}_2(q)/S = \{kS : k \in K\}$. Note that an element kS , with $k \in K$, can be identified with the element of order 3 given by khk^{-1} .

4. Outline of the proof of Theorem 1.3

Let $X = (V, E)$ be a cubic arc-transitive graph satisfying Hypothesis 1. Then, we have that $G^* = \text{PGL}_2(q)$ and $G = \text{PSL}_2(q)$, for some odd prime power q . Recall that $G_v = \langle h \rangle$ and $G_v^* = \langle h, r \rangle$. Given an intersecting set $\mathcal{F} \subset G$, we have seen that we may assume $1 \in \mathcal{F}$. Therefore, every non-identity element of \mathcal{F} is conjugate to h . If $|\mathcal{F}| \geq 2$, then it contains an

element x of order 3, and there exists $g \in G$ such that $gxg^{-1} \in \{h, h^{-1}\}$. We may also assume that \mathcal{F} contains h or h^{-1} , provided that $|\mathcal{F}| \geq 2$, as we can replace \mathcal{F} with the set $g\mathcal{F}g^{-1}$. Therefore, we will always assume that

$$\text{any intersecting set } \mathcal{F} \subset G \text{ such that } |\mathcal{F}| \geq 2 \text{ contains } 1 \text{ and } h. \quad (4.1)$$

We will now give a sketch of the proof of Theorem 1.3. The main result is the intersection density of G when $q = p^k \equiv 2 \pmod{3}$. Note that since $q = p^k \equiv 2 \pmod{3}$, $p \equiv 2 \pmod{3}$ and k is odd.

We will use the fact that all elements of order 3 in G^* , that are conjugate to elements of $G_v = \langle h \rangle$, must be contained in G . Provided that $p \neq 3$, there is only one conjugacy class of elements of order 3 in G , so the elements h and h^{-1} are conjugate in G , and it follows that the set of all the elements of G of order 3, which we denote by \mathcal{C}_3 is a single conjugacy class in G .

By definition, in the derangement graph Γ_G of G , two vertices $x, y \in G$ are not adjacent in Γ_G if and only if xy^{-1} is conjugate to an element in G_v . So a clique in $\overline{\Gamma_G}$ is an intersecting set of permutations. Hence, \mathcal{C}_3 is the neighbourhood of the identity permutation in $\overline{\Gamma_G}$, and by (4.1), any maximum clique of $\overline{\Gamma_G}$ is contained in $\{1\} \cup \mathcal{C}_3$, and, provided it is of size at least 2, contains h . Now, let Γ be the first subconstituent of $\overline{\Gamma_G}$, that is, $\Gamma = \overline{\Gamma_G}[\mathcal{C}_3]$ is the subgraph of $\overline{\Gamma_G}$ induced by \mathcal{C}_3 . It is clear that

$$\rho(G) = \frac{\omega(\overline{\Gamma_G})}{|G|/|V|} = \frac{1 + \omega(\Gamma)}{|G|/|V|}. \quad (4.2)$$

The analysis now entirely depends on the first subconstituent of Γ . The main step of the proof is the next lemma that we prove in the following section.

Lemma 4.1. *Assume that $q = p^k$, where p is odd and $p \neq 3$ and let $\Gamma = \overline{\Gamma_G}[\mathcal{C}_3]$ as above.*

1. $G^* \leq \text{Aut}(\Gamma)$ and acts transitively by conjugation. The vertex stabilizer with this action is isomorphic to $C_{G^*}(h)$.
2. If $q \equiv 2 \pmod{3}$, then Γ is either a perfect matching or the union of two orbital graphs of G^* (these orbital graphs correspond to the suborbits $\{h^{-1}\}$ and a set N , with $|N| = q + 1$).
3. If $q \equiv 2 \pmod{3}$, then the subgraph of Γ induced by N is a Cayley graph on \mathbb{Z}_{q+1} .

Since Γ is vertex transitive, all first subconstituents are isomorphic. Let Δ be the neighbourhood of h in Γ . Note that $h^{-1} \in \Delta$. Define $\tilde{\Gamma} = \Gamma[\Delta \setminus \{h^{-1}\}]$. We will show that $\tilde{\Gamma}$ is either the empty graph, a union of cycles of length at least 4, or a perfect matching. From this, we completely determine all possible values of $\omega(\Gamma)$, and therefore, $\rho(G)$.

Then we consider the final cases for $\rho(G^*)$. By Lemma in [MRS21], we know that $\rho(G^*) \leq \rho(G)$. If $\rho(G) = 1$, then it is clear that $\rho(G^*) = 1$. If $\rho(G) > 1$, then we also show that $\rho(G^*) = 1$. To do this, we consider $\mathcal{F} \subset G^*$ an intersecting set of maximum size, then look at the structure of $|\mathcal{F} \cap G|$.

5. Proof of Lemma 4.1

Let X, G, G^*, p , and k be as in Hypothesis 1. Recall that \mathcal{C}_3 is the conjugacy class of elements of order three, $\Gamma = \overline{\Gamma_G}[\mathcal{C}_3]$, and $\tilde{\Gamma} = \Gamma[\Delta]$, where $\Delta = N_\Gamma(h)$. Assume that $p \neq 3$, so $q \equiv \pm 1 \pmod{3}$.

5.1. Proof of Lemma 4.1(1)

Since the vertices of Γ form a single conjugacy class in G^* , it is clear that G^* acts transitively on the vertices of Γ by conjugation. To prove that Γ is vertex transitive, it is enough to show that conjugation is an automorphism of Γ . Recall that two elements g and g' of \mathcal{C}_3 are adjacent if and only if $g'g^{-1} \in \mathcal{C}_3$. If $g, g' \in \mathcal{C}_3$ and $x \in G^*$, then using the fact that \mathcal{C}_3 is a conjugacy class, we have

$$(xg'x^{-1})(xg^{-1}x^{-1}) \in \mathcal{C}_3 \iff g'g^{-1} \in \mathcal{C}_3.$$

This proves that the transitive action of G^* on the vertices of Γ by conjugation preserves the edges and non-edges of Γ . If the kernel of this action is non-trivial, then there exists a non-trivial element $x \in G^*$ that commutes with every element of \mathcal{C}_3 . As $p \neq 3$, every subgroup of order 3 is self-centralizing, so x belongs to every subgroup of order 3 of G and can only be the identity. This implies that the kernel of the action of G^* is trivial, or equivalently, G^* acts faithfully.

Hence, $G^* \leq \text{Aut}(\Gamma)$, acting by conjugation, is a transitive subgroup, and thus Γ is vertex transitive. The point-stabilizer of this transitive action is the subgroup $C_{G^*}(h)$, which is a cyclic group of order $q + 1$. By the orbit stabilizer lemma, the cosets of $C_{G^*}(h)$ in G are in one-to-one correspondence with elements of \mathcal{C}_3 , so we can either think about cosets of $C_{G^*}(h)$ in G , or elements of \mathcal{C}_3 . In particular, the correspondence is given by

$$ghg^{-1} \leftrightarrow gC_{G^*}(h). \tag{5.1}$$

This completes the proof of (1).

5.2. Proof of Lemma 4.1(2)

Assume $q \equiv 2 \pmod{3}$ and q odd. By Lemma 2.1, Γ is a union of basic orbital graphs of G^* . Let $\Delta = N_\Gamma(h)$. Then, Δ is a union of suborbits. As h and h^{-1} are adjacent in Γ and $\{h^{-1}\}$ is an orbit of $C_{G^*}(h)$, it is clear that $\{h^{-1}\}$ is among these suborbits. If Γ is equal to the orbital graph corresponding to the suborbit $\{h^{-1}\}$, then it is a perfect matching. So we may assume that Γ is not a perfect matching, implying that $\Delta \setminus \{h^{-1}\} \neq \emptyset$. We define the set $N := \Delta \setminus \{h^{-1}\}$. We claim that N is an orbit of $C_{G^*}(h)$ of size $q + 1$.

First we prove that $|N| \geq q + 1$. As $p \neq 3$, every subgroup of order 3 is self-centralizing, so for any $h' \in \mathcal{C}_3$ such that $h' \notin \langle h \rangle$, we have $C_{G^*}(h) \cap C_{G^*}(h') = \{1\}$. Let $x \in N$. If $|\{g x g^{-1} : g \in C_{G^*}(h)\}| < q + 1$, then there exist $g, g' \in C_{G^*}(h)$ such that $g x g^{-1} = g' x (g')^{-1}$. But, this implies $g^{-1}g' \in C_{G^*}(x)$, since we have $g, g' \in C_{G^*}(h)$ and $C_{G^*}(h) \cap C_{G^*}(x) = \{1\}$, we know that $g = g'$. Consequently, $|\{g x g^{-1} : g \in C_{G^*}(h)\}| = q + 1$, and so $|N| \geq q + 1$.

Next, we prove that $|N| \leq q + 1$. As $q \equiv 2 \pmod{3}$, there is a unique conjugacy class of subgroups of order 3 in G . We set $H = \langle h \rangle$ with

$$h = \overline{\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}}.$$

Recall that the subgroup K defined in (3.2) is a transversal of $S = C_{G^*}(h)$. In (5.1), the element $x = ghg^{-1} \in N$ corresponds to the coset $gC_{G^*}(h)$. By the fact that K is a transversal of $C_{G^*}(h)$ in G^* , we may choose $g = k$ to be in K so that $x = ghg^{-1} = khk^{-1} \in N$. Hence, we consider the non-identity element

$$k = \overline{\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}} \in K,$$

that is, $(a, b) \neq (0, 1)$. Now, we have

$$\overline{\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}^{-1}}^{-1} = \overline{\begin{bmatrix} a & -b^{-1}(a^2 + a + 1) \\ b & -1 - a \end{bmatrix}}. \quad (5.2)$$

In addition, we have

$$\overline{\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^{-1}}^{-1} = \overline{\begin{bmatrix} -a + b^{-1}(a^2 + a + 1) & a \\ a - b + 1 & b \end{bmatrix}}. \quad (5.3)$$

The above expressions will be useful due to the fact that ghg^{-1} and h are adjacent if and only if $ghg^{-1}h^{-1}$ is an element of order 3. By (5.3),

$$\overline{\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}^{-1}}^{-1} \quad \text{and} \quad \overline{\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}}$$

are adjacent if and only if $b - a + b^{-1}(a^2 + a + 1) = \pm 1$. We can reformulate this as

$$b^2 - ab + a^2 + a + 1 = \pm b. \quad (5.4)$$

If $b^2 - ab + a^2 + a + 1 = -b$, then we have $(b + (1 - a)2^{-1})^2 - (1 - a)^2 4^{-1} + a^2 + a + 1 = 0$. Hence, if $z = b + (1 - a)2^{-1}$, then

$$z^2 + (1 - 4^{-1})a^2 + (2^{-1} + 1)a + (1 - 4^{-1}) = 0. \quad (5.5)$$

By letting $X = 2z$ and $Y = a + 1$, we can see that (5.5) becomes

$$X^2 + 3Y^2 = 0. \quad (5.6)$$

As -3 is not a square whenever $q \equiv 2 \pmod{3}$, we conclude that (5.6) has a unique solution, given by $(X, Y) = (0, 0)$, or equivalently, $a = b = -1$. In particular, we have

$$\overline{\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}^{-1}}^{-1} = \overline{\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}}.$$

Therefore, this solution corresponds to h^{-1} .

If $b^2 - ab + a^2 + a + 1 = b$, then

$$b^2 - (a + 1)b + a^2 + a + 1 = (b - (a + 1)2^{-1})^2 - 4^{-1}(a + 1)^2 + a^2 + a + 1 = 0.$$

Setting $z = b - (a + 1)2^{-1}$, this is

$$z^2 + (1 - 4^{-1})a^2 + (1 - 2^{-1})a + (1 - 4^{-1}) = 0, \tag{5.7}$$

which is equivalent to

$$4z^2 + 3a^2 + 2a + 3 = 0. \tag{5.8}$$

Similar to the previous case, we set $X = 2z$, $Y = a + 3^{-1}$, and $\gamma = 1 - 3^{-2}$ and (5.8) becomes

$$X^2 + 3Y^2 + 3\gamma = 0. \tag{5.9}$$

From [LN97, Lemma 6.24], the above equation has exactly $q + 1$ solutions in \mathbb{F}_q , as -3 is not a square in \mathbb{F}_q . Each of these $q + 1$ solutions produce a unique element of N . Therefore, $|N| = q + 1$. This completes the proof.

5.3. Proof of Lemma 4.1 (3)

We saw in the previous section that if $q \equiv 2 \pmod{3}$, then Γ is a union of at most two orbital graphs. In particular, the neighbourhood of h in Γ is either $\Delta = \{h^{-1}\}$ or $\Delta = \{h^{-1}\} \cup N$, where N is a suborbit of size $|N| = q + 1$.

If $\Delta = \{h^{-1}\}$, then the statement in Lemma 4.1 (3) holds. If $|N| = q + 1$, then $C_{G^*}(h)$ acts regularly on N , and so the subgraph of Γ induced by N is a Cayley graph of $C_{G^*}(h) \cong \mathbb{Z}_{q+1}$.

Let $\tilde{\Gamma}$ be the subgraph of Γ induced by N . Assume that $C_{G^*}(h) = \langle V \rangle \cong \mathbb{Z}_{q+1}$. If $U \in N$, then the vertices of $\tilde{\Gamma}$ are V^iUV^{-i} for $0 \leq i \leq q$, and a vertex is adjacent to U if and only if $V^iUV^{-i}U^{-1}$ has order 3. Thus $\tilde{\Gamma}$ is isomorphic to $\text{Cay}(\langle V \rangle, T)$, where

$$T = \{V^i : 0 \leq i \leq q, \text{Tr}(V^iUV^{-i}U^{-1}) = \pm 1\}.$$

As the trace is invariant by cyclic permutation, we also have

$$\text{Tr}(V^iUV^{-i}U^{-1}) = \text{Tr}(UV^{-i}U^{-1}V^i) = \text{Tr}\left(U(V^{-i}UV^i)^{-1}\right)$$

for any $0 \leq i \leq q$.

6. The 1-arc regular subgroup $G = \text{PSL}_2(q)$

Let X, G, G^* be as in Hypothesis 1 and $q \equiv 2 \pmod{3}$. Let Γ be the subgraph of $\overline{\Gamma_G}$ induced by \mathcal{C}_3 . Under the assumption in (4.1), a maximum clique of Γ contains h , so under this assumption, a canonical coclique of Γ_G is $\langle h \rangle$; in Γ a **canonical clique** is a clique equal to $\{h, h^{-1}\}$. We recall the following lemma.

Lemma 6.1. [HKKM25, Proposition 2.13] *Let $q = p^k$ for a prime $p \neq 3$. Then, under the assumption (4.1), the non-canonical cocliques of Γ_G do not contain the vertex-stabilizer $G_v = \langle h \rangle$.*

By Lemma 4.1, Γ is equal to either the orbital graph of corresponding to the suborbit $\{h^{-1}\}$ or the union of two orbital graphs corresponding to the suborbits $\{h^{-1}\}$ and N , where $|N| = q + 1$. If Γ is the orbital graph corresponding to $\{h^{-1}\}$, then it is equal to a perfect matching. Therefore, $\omega(\Gamma) = 2$, and by (4.2), we have $\rho(G) = 1$.

If the neighbourhood of h in Γ is $\Delta = \{h^{-1}\} \cup N$, where N is a suborbit of $C_{G^*}(h)$, then by Lemma 4.1 (3), we know that $\tilde{\Gamma} = \Gamma[N]$ is a Cayley graph of $C_{G^*}(h)$. Under the assumption (4.1), Lemma 6.1 implies that a non-canonical clique of Γ cannot contain h^{-1} . This means that we can focus on $\omega(\tilde{\Gamma})$, since in this case

$$\rho(G) = \frac{2 + \omega(\tilde{\Gamma})}{|G|/|V|}.$$

Let $U \in N$, from Equation (5.2), and the comments following it,

$$U = \overline{\begin{bmatrix} a & b - a - 1 \\ b & -1 - a \end{bmatrix}} \text{ and } U^{-1} = \overline{\begin{bmatrix} -1 - a & -b + a + 1 \\ -b & a \end{bmatrix}}$$

for some $a \in \mathbb{F}_q$ and $b \in \mathbb{F}_q^*$ that satisfy $b^2 - ab + a^2 + a + 1 = b$. As N is an orbit of $C_{G^*}(h)$ acting by conjugation, we can obtain all elements of N by conjugating U . Recall that

$$C_{G^*}(h) = \left\{ \overline{\begin{bmatrix} 1 & \alpha \\ -\alpha & 1 + \alpha \end{bmatrix}} : \alpha \in \mathbb{F}_q \right\} \cup \left\{ \overline{\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}} \right\}. \quad (6.1)$$

Assume that $C_{G^*}(h) = \langle V \rangle$, where $V = \overline{\begin{bmatrix} 1 & \alpha' \\ -\alpha' & 1 + \alpha' \end{bmatrix}}$, for some $\alpha' \in \mathbb{F}_q$ and $V^{\frac{q+1}{3}} = h$. From this, we can see that $N = \{V^i U V^{-i} : 0 \leq i \leq q\}$. The next results consider the adjacencies between the elements $V^i U V^{-i}$.

Lemma 6.2. *For any $i \in \{1, 2\}$, the vertices U and $h^i U h^{-i}$ are not adjacent in $\tilde{\Gamma}$.*

Proof. We note that

$$\begin{aligned} h U h^{-1} U^{-1} &= \overline{\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b - a - 1 \\ b & -1 - a \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 - a & -b + a + 1 \\ -b & a \end{bmatrix}} \\ &= \overline{\begin{bmatrix} a^2 - (a+1)b + b^2 + 2a + 1 & -a^2 + (a+2)b - b^2 - 2a - 1 \\ a^2 - ab + b^2 + a & -a \end{bmatrix}} \\ &= \overline{\begin{bmatrix} a & -a + b \\ b - 1 & -a \end{bmatrix}}. \end{aligned}$$

Hence, the trace of $h U h^{-1} U^{-1}$ is equal to 0, so U and $h U h^{-1}$ cannot be adjacent. Since the trace is invariant by cyclic permutation, we have

$$0 = \text{Tr}(h U h^{-1} U^{-1}) = \text{Tr}(U h^{-1} U^{-1} h) = \text{Tr}(U (h^{-1} U h)^{-1}).$$

Therefore, U is also not adjacent to $h^{-1} U h = h^2 U h^{-2}$. \square

Now, we consider the adjacency between U and V^iUV^{-i} , where $i \neq \pm \frac{q+1}{3}$. Let $i \in \{0, \dots, q\} \setminus \{\frac{q+1}{3}, \frac{2(q+1)}{3}\}$. Then, there exists $\alpha \in \mathbb{F}_q$ such that

$$V^i = \overline{\begin{bmatrix} 1 & \alpha \\ -\alpha & 1 + \alpha \end{bmatrix}}.$$

Note that the inverse of this element is

$$(\alpha^2 + \alpha + 1)^{-1} \overline{\begin{bmatrix} 1 + \alpha & -\alpha \\ \alpha & 1 \end{bmatrix}}.$$

Lemma 6.3. *The trace of $V^iUV^{-i}U^{-1}$ is $2(\alpha + 1)(\alpha^2 + \alpha + 1)^{-1}$.*

Proof. We have

$$\begin{aligned} & V^iUV^{-i}U^{-1} \\ &= (\alpha^2 + \alpha + 1)^{-1} \overline{\begin{bmatrix} 1 & \alpha \\ -\alpha & 1 + \alpha \end{bmatrix} \begin{bmatrix} a & b - a - 1 \\ b & -1 - a \end{bmatrix} \begin{bmatrix} 1 + \alpha & -\alpha \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} -1 - a & -b + a + 1 \\ -b & a \end{bmatrix}} \\ &= (\alpha^2 + \alpha + 1)^{-1} \overline{\begin{bmatrix} a\alpha^2 + (a - b + 1)\alpha + 1 & * \\ * & -a\alpha^2 - (a - b - 1)\alpha + 1 \end{bmatrix}}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{Tr}(V^iUV^{-i}U^{-1}) &= (\alpha^2 + \alpha + 1)^{-1}(a\alpha^2 + (a - b + 1)\alpha + 1 - a\alpha^2 - (a - b - 1)\alpha + 1) \\ &= 2(\alpha + 1)(\alpha^2 + \alpha + 1)^{-1}. \quad \square \end{aligned}$$

The following corollary is immediate.

Corollary 6.4. *The vertices U and V^iUV^{-i} are adjacent if $2(\alpha + 1) = \pm(\alpha^2 + \alpha + 1)$.*

The next theorem is the main result of this section.

Theorem 6.5. *If $q \equiv 2 \pmod{3}$, then*

$$\rho(G) = \begin{cases} \frac{4}{3} & \text{if } q = 5^{2k+1}, \text{ for some integer } k \geq 0, \\ 1 & \text{if } q \equiv \pm 2 \pmod{5}, \\ \frac{4}{3} & \text{if } q \equiv \pm 1 \pmod{5}. \end{cases} \quad (6.2)$$

Proof. From Lemma 4.1, the graph $\tilde{\Gamma}$ is a Cayley graph of $C_{G^*}(h) = \langle V \rangle$. Using Corollary 6.4, we will show that the degree of $\tilde{\Gamma}$ is either 0, 1 or 2, by finding the number of solutions to $2(\alpha + 1) = \pm(\alpha^2 + \alpha + 1)$.

First consider the equation $2(\alpha + 1) = -(\alpha^2 + \alpha + 1)$, in this case $\alpha^2 + 3\alpha + 3 = 0$ and

$$(2\alpha + 3)^2 = -3.$$

This has no solutions, since -3 is never a square in \mathbb{F}_q when $q \equiv 2 \pmod{3}$.

Next consider the other equation, $2(\alpha + 1) = (\alpha^2 + \alpha + 1)$. In this case $\alpha^2 - \alpha - 1 = 0$ and we have

$$(2\alpha - 1)^2 = 5. \tag{6.3}$$

Since the above equation is quadratic in α and has coefficients in \mathbb{F}_p , its solutions are in \mathbb{F}_{p^2} . As q is an odd power of p (this follows from $q \equiv 2 \pmod{3}$), \mathbb{F}_{p^2} is not a subfield of \mathbb{F}_q , and so it is clear that (6.3) has a solution in \mathbb{F}_q if and only if it has a solution in \mathbb{F}_p .

First consider the case where $p = 5$, then $q = 5^{2k+1}$ for some $k \geq 0$. Equation (6.3) becomes $2\alpha - 1 = 0$, so there is only one solution, namely $\alpha = 2^{-1} = 2$. In this case, the degree of $\tilde{\Gamma}$ is 1, so it is a union of edges. Therefore,

$$\rho(G) = \frac{2 + \omega(\tilde{\Gamma})}{|G|/|V|} = \frac{4}{3}.$$

Next, assume that $p \neq 5$. Using the Legendre symbol, since $p \neq 5$, we have

$$\left(\frac{5}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{5}, \\ -1 & \text{if } p \equiv \pm 2 \pmod{5}. \end{cases}$$

Hence, 5 is a square in \mathbb{F}_p if and only if $p \equiv \pm 1 \pmod{5}$.

If $p \equiv \pm 2 \pmod{5}$, then $(2\alpha - 1)^2 = 5$ has no solutions and $\tilde{\Gamma}$ is a coclique. Therefore, in this case

$$\rho(G) = \frac{2 + \omega(\tilde{\Gamma})}{|G|/|V|} = \frac{3}{3} = 1.$$

Finally, if $p \equiv \pm 1 \pmod{5}$, then $(2\alpha - 1)^2 = 5$ has exactly 2 solutions and U has valency 2 in $\tilde{\Gamma}$, implying $\tilde{\Gamma}$ is a union of cycles. We will show that these cycles can never have length 3. Indeed, if a component of $\tilde{\Gamma}$ is a cycle of length 3, then all components of $\tilde{\Gamma}$ are cycles of length 3 due to fact that $\tilde{\Gamma}$ is a Cayley graph. In particular, the vertices in the component containing U would be exactly $U, V^{\frac{q+1}{3}}UV^{-\frac{q+1}{3}} = hUh^{-1}$, and $V^{\frac{2(q+1)}{3}}UV^{-\frac{2(q+1)}{3}} = h^{-1}Uh$. But by Lemma 6.2, we obtain a contradiction. Hence, the components of $\tilde{\Gamma}$ are cycles of length at least 4. Therefore,

$$\rho(G) = \frac{2 + \omega(\tilde{\Gamma})}{|G|/|V|} = \frac{4}{3}.$$

This completes the proof. □

7. The 2-arc regular group $G^* = \text{PGL}_2(q)$

Again, we let X, G, G^* be as in Hypothesis 1. In this section, we determine all the possible intersection densities of $\rho(G^*)$. We first recall that $\rho(G^*) \leq \rho(G)$. By Theorem 6.5, we conclude that $\rho(G^*) = 1$, whenever $p \equiv \pm 2 \pmod{5}$. Therefore, we will assume that $p \equiv a \pmod{5}$, where $a \in \{0, \pm 1\}$.

We recall from Theorem 6.5 that $\alpha(\Gamma_G) \leq 4$. As $G \leq G^*$ and $r \in G^* \setminus G$ is an involution, we know that $G^* = G \rtimes \langle r \rangle$. Let $\mathcal{F} \subset G^*$ be an intersecting set. Then, it is clear that $|\mathcal{F} \cap G| \leq 4$ and $|\mathcal{F} \cap Gr| \leq 4$. Consequently, we deduce $|\mathcal{F}| \leq 8$ and thus $\rho(G^*) \in \{1, \frac{7}{6}, \frac{4}{3}\}$. We will show now that $\rho(G^*) = 1$.

Recall from Lemma 3.4, if $\bar{A} \in G^*$ is an involution fixing a vertex, then $\text{Tr}(A) = 0$.

Theorem 7.1. *If $p \equiv 0, \pm 1 \pmod{5}$, then $\rho(G^*) = 1$.*

Proof. Without loss of generality, let \mathcal{F} be an intersecting set of G^* . We note that the vertex-stabilizer $G_v^* = \langle h, r \rangle$ has order 6, so $\alpha(\Gamma_G^*) \geq 6$. Let $\mathcal{F} \subset G^*$ be an intersecting set. We will show by contradiction that if $|\mathcal{F} \cap Gr| = 4$, then $|\mathcal{F} \cap G| \leq 1$.

Assume that $|\mathcal{F} \cap Gr| = 4$ and $|\mathcal{F} \cap G| \geq 2$. As $|\mathcal{F} \cap G| \geq 2$, we may further assume that \mathcal{F} contains 1 (multiply \mathcal{F} by the inverse of an element from $\mathcal{F} \cap G$, so that, we still assume without loss of generality that $|\mathcal{F} \cap Gr| = 4$ and $|\mathcal{F} \cap G| \geq 2$.) Further, by taking a conjugate, we also assume $h \in \mathcal{F}$. Let $\mathcal{F} \cap Gr = \{X, Y, Z, W\}$. Since X, Y, Z, W fix a vertex (equivalently, are adjacent to 1), by Lemma 3.4, we must have

$$X = \overline{\begin{bmatrix} x & t_x \\ r_x & -x \end{bmatrix}}, Y = \overline{\begin{bmatrix} y & t_y \\ r_y & -y \end{bmatrix}}, Z = \overline{\begin{bmatrix} z & t_z \\ r_z & -z \end{bmatrix}}, W = \overline{\begin{bmatrix} w & t_w \\ r_w & -w \end{bmatrix}},$$

for some $x, y, z, w \in \mathbb{F}_q$ and $r_u, t_u \in \mathbb{F}_q$ for $u \in \{x, y, z, w\}$. Moreover, since X is also adjacent to h in $\overline{\Gamma_{G^*}}$, using the fact that $h \in G$ and $X \in G^* \setminus G$, we have $\text{Tr}(Xh) = 0$. By noting that

$$Xh = \overline{\begin{bmatrix} t_x & * \\ * & x - r_x \end{bmatrix}},$$

we deduce that $r_x = t_x + x$. Hence, we have

$$X = \overline{\begin{bmatrix} x & t_x \\ t_x + x & -x \end{bmatrix}}.$$

Similarly, we also have

$$Y = \overline{\begin{bmatrix} y & t_y \\ t_y + y & -y \end{bmatrix}}, Z = \overline{\begin{bmatrix} z & t_z \\ t_z + z & -z \end{bmatrix}}, W = \overline{\begin{bmatrix} w & t_w \\ t_w + w & -w \end{bmatrix}}.$$

Claim 7.2. *One of x, y, z, w is equal to 0.*

Proof of Claim 7.2. If x, y, z, w are all non-zero, then we have

$$X = \overline{\begin{bmatrix} 1 & a_x \\ a_x + 1 & -1 \end{bmatrix}}, Y = \overline{\begin{bmatrix} 1 & a_y \\ a_y + 1 & -1 \end{bmatrix}}, Z = \overline{\begin{bmatrix} 1 & a_z \\ a_z + 1 & -1 \end{bmatrix}}, W = \overline{\begin{bmatrix} 1 & a_w \\ a_w + 1 & -1 \end{bmatrix}}.$$

As X and Y are adjacent in $\overline{\Gamma_{G^*}}$, we have $\text{Tr}(YX) = \pm 1$. By noting that

$$YX = \overline{\begin{bmatrix} 1 + a_x a_y + a_x & * \\ * & 1 + a_x a_y + a_y \end{bmatrix}},$$

we conclude, $2 + 2a_x a_y + a_x + a_y = \pm 1$. Similarly, we can show that

$$\begin{aligned} 2 + 2a_x a_z + a_x + a_z &= \pm 1 \\ 2 + 2a_x a_w + a_x + a_w &= \pm 1. \end{aligned}$$

Since these three equations are equal to 1 or -1 , without loss of generality we may assume that $2 + 2a_x a_y + a_x + a_y = 2 + 2a_x a_z + a_x + a_z$. This implies that

$$(2a_x + 1)a_y = (2a_x + 1)a_z. \quad (7.1)$$

If $a_x \neq -2^{-1}$, then $a_y = a_z$ which means $Y = Z$, so we must have $a_x = -2^{-1}$. In this case we have

$$2 + 2a_x a_z + a_x + a_z = 2 - 2^{-1}.$$

We know that $2 - 2^{-1} \neq 1$, so $2 - 2^{-1} = -1$. The latter happens if and only if $p = 5$. Therefore, if $p \neq 5$, then (7.1) does not hold which is a contradiction with X adjacent to Y . So one of x, y, z, w must be equal to 0.

Consider the case when $p = 5$ and $a_x = -2^{-1} = 2$, then none of a_y, a_z, a_w can also be equal to 2. The adjacency between Y, Z , and W yield the equations

$$\begin{aligned} 2 + a_y + a_z + 2a_y a_z &= \pm 1 \\ 2 + a_y + a_w + 2a_y a_w &= \pm 1 \\ 2 + a_z + a_w + 2a_z a_w &= \pm 1. \end{aligned}$$

Again, two of these equations must be both equal to 1 or -1 , so we may assume without loss of generality that $2 + a_y + a_z + 2a_y a_z = 2 + a_y + a_w + 2a_y a_w$. Therefore, we have

$$(2a_y + 1)a_z = (2a_y + 1)a_w.$$

As $a_y \neq 2$, we must have $a_z = a_w$ or equivalently $Z = W$. This is a contradiction and we conclude that exactly one of x, y, z, w is equal to 0. \square

From Claim 7.2, we may assume without loss of generality that $x = 0$, and so

$$X = \overline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}.$$

Claim 7.3. X cannot be adjacent to $Y, Z,$ and W .

Proof of Claim 7.3. Clearly, we have

$$\begin{aligned}\operatorname{Tr}(XY) &= 2a_y + 1 = \pm 1 \\ \operatorname{Tr}(XZ) &= 2a_z + 1 = \pm 1 \\ \operatorname{Tr}(XW) &= 2a_w + 1 = \pm 1.\end{aligned}$$

We deduce that a_y, a_z, a_w are not distinct, implying that Y, Z, W are not distinct, which is a contradiction. \square

We deduce from Claim 7.3 that $|\mathcal{F} \cap Gr| \leq 3$, contradicting the assumption that $|\mathcal{F} \cap Gr| = 4$. Consequently, if $|\mathcal{F} \cap Gr| = 4$, then $|\mathcal{F} \cap G| \leq 1$. This shows that any intersecting set \mathcal{F} of G^* of maximum size must have the property that $|\mathcal{F}| = 6$, and $|\mathcal{F} \cap G| = |\mathcal{F} \cap Gr| = 3$. This completes the proof. \square

8. Appendix

The following graphs (using Conder's naming system) satisfy Hypothesis 1: C4.1, C84.1, C120.2, C220.2, C364.5, C364.6, C680.1, C816.1, C816.5, C816.6, C816.7, C1140.1, C1140.6, C1140.8, C1140.9, C2024.3, C2024.4, C2024.5, C2024.9, C2024.11, C2024.12, C2600.3, C2600.4, C4060.1, C4060.2, C4060.3, C4060.4, C4060.5, C4060.6, C4060.10, C4060.11, C4060.14, C4060.15, C4960.11, C4960.12, C4960.14, C4960.15, C5000.1, C5456.1, C5456.2, C5456.3, C8436.1, C8436.2, C8436.3, C8436.4, C8436.5, C8436.6, C8436.7, C8436.8, C8436.13, C8436.15, C8436.17, C8436.18, C8436.20, C8436.21, C8436.23, C8436.24.

Acknowledgements

We wish to thank the anonymous referees who gave valuable comments on our manuscript.

References

- [BMR24] Angelot Behajaina, Roghayeh Maleki, and Andriaherimanana Sarobidy Razafima-hatratra. Intersection density of imprimitive groups of degree pq . *J. Combin. Theory Ser. A*, 208:Paper No. 105922, 23, 2024. doi:10.1016/j.jcta.2024.105922.
- [CGS25] M. Cazzola, L. Gogniat, and P. Spiga. Kronecker classes and cliques in derange-ment graphs. 2025. arXiv:2502.01287.
- [Con25] Marston Conder. Trivalent (cubic) symmetric graphs on up to 10000 vertices, Accessed July 3, 2025. URL: <https://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt>.

- [COTR06] P. J. Cameron, G. R. Omidi, and B. Tayfeh-Rezaie. 3-designs from $\text{PGL}(2, q)$. *Electron. J. Combin.*, 13(1):Research Paper 50, 11, 2006. doi:10.37236/1076.
- [DMM22] Ted Dobson, Aleksander Malnič, and Dragan Marušič. *Symmetry in graphs*, volume 198 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2022. doi:10.1017/9781108553995.
- [FD77] Péter Frankl and Mikhail Deza. On the maximum number of permutations with given maximal or minimal distance. *J. Combinatorial Theory Ser. A*, 22(3):352–360, 1977. doi:10.1016/0097-3165(77)90009-7.
- [GM09] Chris Godsil and Karen Meagher. A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations. *European J. Combin.*, 30(2):404–414, 2009. doi:10.1016/j.ejc.2008.05.006.
- [HKKM25] Ademir Hujdurović, István Kovács, Klavdija Kutnar, and Dragan Marušič. Intersection density of transitive groups with small cyclic point stabilizers. *European J. Combin.*, 124:Paper No. 104079, 13, 2025. doi:10.1016/j.ejc.2024.104079.
- [HKMM22] Ademir Hujdurović, Klavdija Kutnar, Dragan Marušič, and Štefko Miklavič. Intersection density of transitive groups of certain degrees. *Algebr. Comb.*, 5(2):289–297, 2022. doi:10.5802/alco.209.
- [KMP23] Klavdija Kutnar, Dragan Marušič, and Cyril Pujol. Intersection density of cubic symmetric graphs. *J. Algebraic Combin.*, 57(4):1313–1326, 2023. doi:10.1007/s10801-023-01228-4.
- [LN97] Rudolf Lidl and Harald Niederreiter. *Finite fields*, volume 20 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1997.
- [LSP20] C. H. Li, S. J. Song, and V. Pantangi. Erdős-Ko-Rado problems for permutation groups. 2020. arXiv:2006.10339.
- [MR21] Karen Meagher and A. Sarobidy Razafimahatratra. The Erdős-Ko-Rado theorem for 2-pointwise and 2-setwise intersecting permutations. *Electron. J. Combin.*, 28(4):Paper No. 4.10, 21, 2021. doi:10.37236/9556.
- [MR24] Karen Meagher and Andriaherimanana Sarobidy Razafimahatratra. On the intersection density of the Kneser graph $K(n, 3)$. *European J. Combin.*, 118:Paper No. 103910, 15, 2024. doi:10.1016/j.ejc.2023.103910.
- [MRS21] Karen Meagher, Andriaherimanana Sarobidy Razafimahatratra, and Pablo Spiga. On triangles in derangement graphs. *J. Combin. Theory Ser. A*, 180:Paper No. 105390, 26, 2021. doi:10.1016/j.jcta.2020.105390.
- [MS11] Karen Meagher and Pablo Spiga. An Erdős-Ko-Rado theorem for the derangement graph of $\text{PGL}(2, q)$ acting on the projective line. *J. Combin. Theory Ser. A*, 118(2):532–544, 2011. doi:10.1016/j.jcta.2010.11.003.
- [MST16] Karen Meagher, Pablo Spiga, and Pham Huu Tiep. An Erdős-Ko-Rado theorem for finite 2-transitive groups. *European J. Combin.*, 55:100–118, 2016. doi:10.1016/j.ejc.2016.02.005.