Discrete Mathematics 348 (2025) 114364

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The intersection density of non-quasiprimitive groups of degree $3p$

Roghayeh Maleki [a,*], Andriaherimanana Sarobidy Razafimahatratra [b]

[a] Department of Mathematics and Statistics, University of Regina, 3737 Wascana Parkway, Regina, SK S4S 0A2, Canada
[b] Fields Institute for Research in Mathematical Sciences, 222 College St Toronto, ON M5T 3J1, Canada

ARTICLE INFO

Article history:
Received 9 August 2024
Received in revised form 18 October 2024
Accepted 5 December 2024
Available online xxxx

Keywords:
Derangement graphs
Cocliques
Projective special linear groups

ABSTRACT

The intersection density of a finite transitive group $G \leq \mathrm{Sym}(\Omega)$ is the rational number $\rho(G)$ given by the ratio between the maximum size of a subset of G in which any two permutations agree on some elements of Ω and the order of a point stabilizer of G. In 2022, Meagher asked whether $\rho(G) \in \{1, \frac{3}{2}, 3\}$ for any transitive group G of degree $3p$, where $p \geq 5$ is an odd prime. If $G \leq \mathrm{Sym}(\Omega)$ is transitive such that $|\Omega| = 3p$, then it is known that $\rho(G) = 1$ whenever (a) G is primitive or (b) G is imprimitive and admits a block of size p or at least two G-invariant partitions of Ω. In order to answer Meagher's question, it is left to analyze the intersection density of groups G admitting a unique G-invariant partition $\mathcal{B}$ whose blocks are of size 3. If G is such a group and $\overline{G}$ is the group induced by the action of G on $\mathcal{B}$, then we denote the kernel of the canonical epimorphism $G \to \overline{G}$ by $\ker(G \to \overline{G})$. The subgroup $\ker(G \to \overline{G})$ is trivial if and only if G is quasiprimitive.

It is shown in this paper that the answer to Meagher's question is affirmative for non-quasiprimitive groups of degree $3p$, unless possibly when $p = q + 1$ is a Fermat prime and Ω admits a unique G-invariant partition $\mathcal{B}$ whose blocks are of size 3 such that the induced action $\overline{G}$ is an almost simple group with socle equal to $\mathrm{PSL}_2(q)$.

1. Introduction

The Erdős-Ko-Rado (EKR) Theorem [8] is a fundamental theorem in extremal set theory. For any two positive integers $n \geq k \geq 1$, we let $\binom{[n]}{k}$ be the collection of all k-subsets of $\{1, 2, \ldots, n\}$. The EKR Theorem is stated as follows.

Theorem 1.1 *(Erdős-Ko-Rado). Let $n \geq k \geq 1$ be two positive integers such that $n \geq 2k$. If $\mathcal{F} \subset \binom{[n]}{k}$ such that $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Moreover, if $n \geq 2k + 1$ then equality holds if and only if $\mathcal{F}$ is the collection of all k-subsets of $\{1, 2, \ldots, n\}$ containing a prescribed element.

* Corresponding author.
E-mail addresses: rmaleki@uregina.ca (R. Maleki), sarobidy@phystech.edu (A.S. Razafimahatratra).

https://doi.org/10.1016/j.disc.2024.114364

The EKR Theorem has been extended to various combinatorial objects throughout the years. See the monograph [11] on EKR type results from an algebraic perspective for more details. This paper is concerned with the extension of the EKR Theorem to transitive permutation groups with fixed degrees.

All groups considered in this paper are finite and all graphs are simple and undirected. Given a transitive group $G \leq \mathrm{Sym}(\Omega)$, we say that $\mathcal{F} \subset G$ is ***intersecting*** if for any $g, h \in \mathcal{F}$, there exists $\omega \in \Omega$ such that $\omega^g = \omega^h$. For any $\omega \in \Omega$, the stabilizer $G_\omega = \{g \in G : \omega^g = \omega\}$ of ω and its cosets are obvious intersecting sets of G. The ***intersection density*** of the transitive group G is then defined to be the rational number

$$\rho(G) := \frac{1}{|G_\omega|} \max\{|\mathcal{F}| : \mathcal{F} \subset G \text{ is intersecting}\}.$$

We note that $\rho(G) \geq 1$ since G_ω itself is intersecting. Transitive groups with intersection density equal to 1 have been subject to a great deal of focus since the paper of Deza and Frankl [5] in the late 70s. More recent results on this topic for instance can be found in [1,4,9,7,10,16,19,23].

Recently, several works on transitive groups with intersection density larger than 1 have appeared in the literature. In particular, the paper [17] by Li, Song and Pantangi, and [20] by Meagher, Spiga and the second author explored the theory of the transitive groups with intersection density larger than 1. In the same paper [20], the following conjectures were posed.

Conjecture 1.2. *Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group.*

(1) If $|\Omega|$ is a prime power, then $\rho(G) = 1$.
(2) If $|\Omega| = 2p$, where p is an odd prime, then $1 \leq \rho(G) \leq 2$.
(3) If $|\Omega| = pq$, where p and q are two odd primes, then $\rho(G) = 1$.

Conjecture 1.2(1) was proved independently in [14] and [17], and Conjecture 1.2(2) was proved in [21]. In [13], Conjecture 1.2(3) was disproved by constructing a family of transitive groups of degree pq, where $p = \frac{q^r-1}{q-1}$, whose intersection density is equal to q. Though the third conjecture was disproved, it is still of interest to know all the possible intersection densities of transitive groups of degree a product of two distinct odd primes. To this end, we recall the following set which was first defined in [20]

$$\mathcal{I}_n := \{\rho(G) : G \leq \mathrm{Sym}(\Omega) \text{ is transitive with } |\Omega| = n\},$$

for $n \geq 2$. Using the computer algebra system `Sagemath` [24], one can for instance verify that $\mathcal{I}_{15} = \{1\}$, whereas $\mathcal{I}_{10} = \{1, 2\}$ and $\mathcal{I}_{39} = \{1, \frac{3}{2}, 3\}$. In general, very little is known about the set $\mathcal{I}_n$ for arbitrary $n \geq 2$. For example, $\mathcal{I}_n = \{1\}$ when n is a power of a prime. In [14], it was shown that $\mathcal{I}_{2p} = \{1, 2\}$, however, there is a gap in the proof as analysis for the quasiprimitive groups (see the definition below) is missing. Though filling this gap seems to be hard, we are inclined to believe that the result in [14] remains correct.

Let $G \leq \mathrm{Sym}(\Omega)$ be a finite transitive group. A G***-invariant partition***, or ***system of imprimitivity***, or a ***complete block system*** of G is a partition that is preserved by the action of G. That is, given a partition $\mathcal{B}$ of Ω, either $B^g = B$ or $B^g \cap B = \varnothing$, for any $B \in \mathcal{B}$ and $g \in G$. The elements of a G-invariant partition of G are called ***blocks***. A G-invariant partition of Ω is trivial if it is one of $\{\Omega\}$ and $\{\{\omega\} : \omega \in \Omega\}$. If the only G-invariant partitions of G are the trivial ones, then we say that G is ***primitive***; otherwise, it is ***imprimitive***. A group $G \leq \mathrm{Sym}(\Omega)$ is ***quasiprimitive*** if any non-trivial normal subgroup of G is transitive. The transitive groups on Ω can therefore be subdivided into certain categories: the quasiprimitive groups, which can be both primitive and imprimitive, and the non-quasiprimitive groups, which are necessarily imprimitive. Let us define the following subsets of $\mathcal{I}_n$, for $n \geq 1$, with respect to these categories

$$\mathcal{P}_n := \{\rho(G) : G \text{ is primitive of degree } n\},$$

$$\mathcal{Q}_n := \{\rho(G) : G \text{ is quasiprimitive and imprimitive of degree } n\},$$

$$\mathcal{NQ}_n := \{\rho(G) : G \text{ is non-quasiprimitive and imprimitive of degree } n\}.$$

In [22], most of the primitive groups of degree a product of two distinct odd primes were shown to have intersection density equal to 1. In [3], some results on the intersection density of imprimitive groups of degree a product of two odd primes were also proved. This paper is a continuation of the larger project whose aim is to determine the exact set $\mathcal{I}_{pq}$ for any two distinct primes p and q. It is usually difficult to determine the set $\mathcal{I}_{pq}$, even for some small values of p and q. Hence, we will restrict ourselves to the study of the set $\mathcal{I}_{3p}$, where $p \geq 5$ is an odd prime. Another motivation for this work is also the following question due to Meagher.

Question 1.3. Let $p \geq 5$ be an odd prime. Is it true that $\mathcal{I}_{3p} \subseteq \{1, \frac{3}{2}, 3\}$?

In [22], it was shown that $\mathcal{P}_{3p} = \{1\}$, for any prime $p \geq 5$. Therefore, we only need to consider the imprimitive groups of degree $3p$. If $G \leq \mathrm{Sym}(\Omega)$ is imprimitive and non-quasiprimitive, then there exists a non-trivial subgroup $N \trianglelefteq G$ which is intransitive. The set $\mathcal{B}$ of orbits of N forms a G-invariant partition of Ω and the induced action, denoted by $\overline{G}_{\mathcal{B}}$, of G on these orbits is transitive of degree p, so it is either solvable or 2-transitive. By [13] and [22], if Ω has more than one non-trivial G-invariant partitions or it admits a G-invariant partitions whose blocks are of size p, then $\rho(G) = 1$. Hence, we can always assume that $\mathcal{B}$ is the unique G-invariant partition of G.

We recall that the socle $\mathrm{Soc}(G)$ of a group G is the subgroup generated by its minimal normal subgroups. The main result of this paper is as follows.

Theorem 1.4. *Let $G \leq \mathrm{Sym}(\Omega)$ be transitive of degree $3p$, $1 \neq N \trianglelefteq G$ be intransitive, and $\overline{G}$ be the induced action of G on the unique G-invariant partitions of Ω consisting of orbits of N. One of the following cases occurs.*

(1) $\overline{G}$ is solvable and $\rho(G) \in \{1, \frac{3}{2}, 3\}$.
(2) $\overline{G}$ is 2-transitive, p is not a Fermat prime, and $\rho(G) = 1$.
(3) $\overline{G}$ is 2-transitive with $\mathrm{Soc}(\overline{G}) \neq \mathrm{PSL}_2(q)$, p is a Fermat prime, and $\rho(G) = 1$.
(4) $\overline{G}$ is an almost simple group whose socle is $\mathrm{PSL}_2(q)$ with $p = q + 1$ a Fermat prime.

In particular, if $p \geq 5$ is not a Fermat prime, then $\mathcal{NQ}_{3p} \subset \{1, \frac{3}{2}, 3\}$.

Strategy of the proof

Our proof relies heavily on a characterization of non-quasiprimitive groups of degree a product of two odd primes with intersection density larger than 1 in [3]. Suppose that $\mathcal{B} = \{B_1, B_2, \ldots, B_p\}$, with $|B_i| = 3$ for all $1 \leq i \leq p$, is the unique G-invariant partition of Ω. We will see that in our case an imprimitive group is non-quasiprimitive if and only if the kernel $\ker(G \to \overline{G})$ of the canonical homomorphism $G \to \overline{G}$ is non-trivial. It was shown in [3] that the only possible groups giving intersection density larger than 1 in the class of non-quasiprimitive imprimitive groups are those with the property that $\ker(G \to \overline{G})$ is derangement-free.

Under this assumption, the analysis can then be divided into two cases, depending on whether $\overline{G}$ is solvable or not. This is due to the famous result of Burnside which asserts that $\overline{G} = \langle \alpha \rangle \rtimes \langle \beta \rangle \leq \mathrm{AGL}_1(p)$ or $\overline{G}$ is 2-transitive. We will show that if $\overline{G}$ is solvable, then the analysis can be further subdivided into two subcases depending on whether $\ker(G \to \overline{G})$ admits an involution or not. If $\ker(G \to \overline{G})$ has no involutions, then the derangement graph of G (see Section 4 for the definition) is a lexicographic product of a graph from a family of graphs defined in Section 3 and an empty graph. From this we can show that the intersection density is in $\{1, \frac{3}{2}, 3\}$. Using the graph in Section 3 along with the No-Homomorphism Lemma, we obtain that the intersection density also belongs to $\{1, \frac{3}{2}, 3\}$ for the case where $\ker(G \to \overline{G})$ has an involution. If $\overline{G}$ is 2-transitive, then we show that G must contain a transitive subgroup H with similar properties[1] to that of G except that $\overline{H}$ is solvable. We obtain an upper bound equal to 1, except when p is a Fermat prime and $\mathrm{Soc}(\overline{G})$ is an almost simple group containing a projective special linear group of degree 2, in which case we only get an upper bound equal to $\frac{3}{2}$.

The proof of Theorem 1.4 then follows from Theorem 7.3, Theorem 7.5, Theorem 8.3, and Theorem 9.2.

Organization of the paper

In Section 2, we give some necessary background results from permutation group theory. In Section 3, we define a family of graphs that are crucial to the proof of the main result. The concept of derangement graphs and some useful results are given in Section 4. In Section 5, we give an analysis of the cases to consider. Then, the solvable case is proved in Section 6, Section 7, and Section 8. In Section 9, we prove the 2-transitive case. We give some open problems regarding the remaining cases in Section 10.

2. Background results on permutation group theory

2.1. Basic notions

Henceforth, we let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group. An (m, n)***-semiregular*** element of G is a permutation that is a product of n cycles of length m. If m and n are clear from the context, then we just use the term ***semiregular*** element. A ***semiregular subgroup*** of G is a subgroup H with the property that for any two elements $\omega, \omega' \in \Omega$, there exists at most one element $h \in H$ such that $\omega' = \omega^h$. A subgroup generated by a semiregular element is clearly a semiregular subgroup.

We say that $G \leq \mathrm{Sym}(\Omega)$ is 2***-transitive*** or ***doubly transitive*** if for any pairs of elements $(\omega_1, \omega_2), (\omega_3, \omega_4) \in \Omega \times \Omega$ such that $\omega_1 \neq \omega_2$ and $\omega_3 \neq \omega_4$, there exists $g \in G$ such that $(\omega_1, \omega_2)^g = (\omega_3, \omega_4)$. In other words, G is transitive on $\Omega^{(2)} = \{(\omega, \omega') \in \Omega \times \Omega : \omega \neq \omega'\}$.

[1] Transitive and $\mathcal{B}$ is its only non-trivial H-invariant partition.

For any G-invariant partition $\mathcal{B}$ of a transitive group $G \leq \mathrm{Sym}(\Omega)$, we may define $\overline{G}_{\mathcal{B}} = \{\overline{g} : g \in G\}$, where $\overline{g}$ is the permutation of $\mathcal{B}$ induced by g. As the groups that we study in this paper admit a unique non-trivial G-invariant partition, we will use the notation $\overline{G}$ instead of $\overline{G}_{\mathcal{B}}$. Clearly, $\overline{G} \leq \mathrm{Sym}(\mathcal{B})$ is transitive. Consequently, G acts on $\mathcal{B}$ via the natural group homomorphisms $G \to \overline{G} \to \mathrm{Sym}(\mathcal{B})$. We define $\ker(G \to \overline{G})$ to be the kernel of the induced action of G on $\mathcal{B}$. By the First Isomorphism Theorem, we have $G/\ker(G \to \overline{G}) \cong \overline{G}$, and so we note that the action of G on $\mathcal{B}$ corresponds to a permutation group of $\mathrm{Sym}(\mathcal{B})$ if and only if $\ker(G \to \overline{G})$ is trivial.

Given $g \in G$ and $B \subset \Omega$ such that $B^g = B$, we let $g_{|B}$ be the restriction of the permutation $g \in \mathrm{Sym}(\Omega)$ onto B. We will denote the order of $g \in G$ by $o(g)$.

2.2. Transitive groups of prime order

Throughout this subsection, we assume that $G \leq \mathrm{Sym}(\Omega)$ is transitive of prime degree p. Recall that the socle of a group is the subgroup generated by its minimal normal subgroups. Let $\mathrm{Soc}(G)$ denote the socle of G.

It was known in the late 1800s due to Burnside that a transitive group of prime degree has to be solvable or 2-transitive. The Classification of Finite Simple Groups (CFSG) made it possible to obtain important classification results in permutation group theory. One of such classifications is that of the 2-transitive groups. A transitive group G of prime degree p which is 2-transitive has to be equal to $\mathrm{AGL}_1(p)$ or an almost simple group, that is, one with the property $\mathrm{Soc}(G) \leq G \leq \mathrm{Aut}(\mathrm{Soc}(G))$, where $\mathrm{Soc}(G)$ is a non-abelian simple group. The possibilities for the socles of G are

(a) $\mathrm{Soc}(G) = C_p$, in which case $G = \mathrm{AGL}_1(p)$,
(b) $\mathrm{Soc}(G) = \mathrm{Alt}(p)$,
(c) $p = 11$ and $G = \mathrm{Soc}(G) = \mathrm{M}_{11}$ or $G = \mathrm{Soc}(G) = \mathrm{PSL}_2(11)$,
(d) $p = 23$ and $G = \mathrm{Soc}(G) = \mathrm{PSL}_2(23)$ or $G = \mathrm{Soc}(G) = \mathrm{M}_{23}$,
(e) $p = \frac{q^n-1}{q-1}$ and $\mathrm{Soc}(G) = \mathrm{PSL}_n(q)$, for some prime power q.

For (e), since $p = \frac{q^n-1}{q-1}$ is a prime, it is not hard to show that in fact n itself is a prime. If $n = 2$, then since $p = q + 1$ is an odd prime, we must have that q is an even power of 2 and p is, therefore, a Fermat prime.

3. Graph theory

Given a graph $X = (V, E)$, we use the notation $x \sim_X y$ to represent the fact that $\{x, y\} \in E$, or equivalently, x and y are adjacent. A ***clique*** in X is a subset of vertices in which any two are adjacent. A ***coclique*** in X is a subset of vertices in which no two are adjacent. The maximum size of a clique and a coclique in a graph X are denoted respectively by $\omega(X)$ and $\alpha(X)$.

Let X and Y be two graphs. The ***lexicographic product*** $X[Y]$ of X and Y (in this order) is the graph with vertex set $V(X) \times V(Y)$ such that two vertices (x, y) and (x', y') in $V(X) \times V(Y)$ are adjacent if and only if

$$\begin{cases} x \sim_X x', \text{ or} \\ x = x' \text{ and } y \sim_Y y'. \end{cases}$$

Let $m, n \geq 2$, k and r be positive integers such that $k \mid r$. For any positive integer t, we let $[t] := \{1, 2, \ldots, t\}$. Define the graph $\Gamma^k_{m,n}(r)$ whose vertex set is

$$V = \{(a, b, c) : a \in [r], b \in [m], c \in [n]\}.$$

Let $\pi = \{P_1, P_2, \ldots, P_k\}$ be a uniform partition of $[r]$, i.e., a partition all of whose parts are of equal size. The edge set of $\Gamma^k_{m,n}(r)$ is defined in a way that

$$(a, b, c) \sim (a', b', c') \Leftrightarrow \begin{cases} c = c' \text{ and } b \neq b', \text{ or} \\ c \neq c' \text{ and } a \text{ and } a' \text{ are in different parts of } \pi. \end{cases}$$

We note that $\Gamma^k_{m,n}(r)$ is independent of the uniform partition π since using a different uniform partition with k blocks yields an isomorphic graph. Hence, we fix a uniform partition $\pi = \{P_1, \ldots, P_k\}$.

Example 3.1. The graph in Fig. 1 is $\Gamma^3_{3,2}(6)$. If the edge between two blobs is black, then the corresponding induced subgraph is $K_{6,6}$, and if it is red, then the induced subgraph is $X[\overline{K}_2]$, where X is the graph defined in Fig. 2.

For any fixed $b \in [m]$ and $c \in [n]$, define $U_{b,c}(r) := \{(a, b, c) : a \in [r]\}$. We note that the set $U_{b,c}(r)$ is a coclique of $\Gamma^k_{m,n}(r)$. Let us now analyze the edges between these cocliques in the graph $\Gamma^k_{m,n}(r)$. We omit the proof of the next proposition since it follows directly from the definition of the edge set.

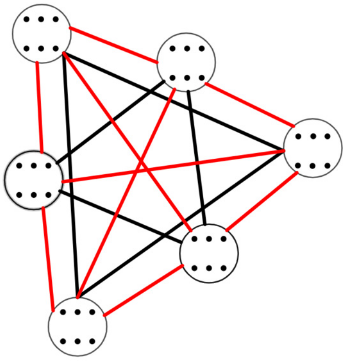

Fig. 1. The graph $\Gamma^3_{3,2}(6)$.

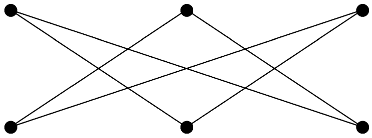

Fig. 2. The graph $K_{3,3}$ minus a perfect matching, which is isomorphic to a cycle of length 6.

Proposition 3.2. *Let $b, b' \in [m]$, and $c, c' \in [n]$ such that $(b, c) \neq (b', c')$. Then, one of the following holds.*

(1) If $c = c'$, then the subgraph of $\Gamma^k_{m,n}(r)$ induced by $U_{b,c}(r) \cup U_{b',c'}(r)$ is a complete bipartite graph $K_{r,r}$.
(2) If $c \neq c'$, then the subgraph of $\Gamma^k_{m,n}(r)$ induced by $U_{b,c}(r) \cup U_{b',c'}(r)$ is the lexicographic product $\widetilde{K}_{k,k}[\overline{K_{\frac{r}{k}}}]$, where the graph $\widetilde{K}_{k,k}$ is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{k,k}$.

The following is an immediate corollary of the above proposition.

Corollary 3.3. *We have the graph isomorphism $\Gamma^k_{m,n}(r) \cong \Gamma^k_{m,n}(k)\left[\overline{K_{\frac{r}{k}}}\right]$, where $\overline{K_{\frac{r}{k}}}$ is the complement of the complete graph $K_{\frac{r}{k}}$.*

Lemma 3.4. *The independence number of $\Gamma^k_{m,n}(k)$ is equal to* $\max\{k, n\}$.

Proof. Let $\mathcal{F}$ be a coclique of $\Gamma^k_{m,n}(k)$. If $\mathcal{F} \subset U_{1,1}(k)$, then $|\mathcal{F}| \leq k$. Thus, we may assume without loss of generality that $\mathcal{F} \cap U_{1,1}(k) \neq \varnothing$ and that $\mathcal{F} \setminus U_{1,1}(k) \neq \varnothing$. Let $z \in \mathcal{F} \cap U_{1,1}(k) = \{(a, 1, 1) : a = j\}$, for some $j \in [k]$. Thus, $z = (j, 1, 1)$. Let us decompose $\mathcal{F}$ into

$$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_n,$$

where $\mathcal{F}_i = \mathcal{F} \cap \left(U_{1,i}(k) \cup U_{2,i}(k) \cup \ldots \cup U_{m,i}(k)\right)$, for any $i \in \{1, 2, \ldots, n\}$. Then, for any $i \in \{2, 3, \ldots, n\}$, the vertex z is non-adjacent to the vertex (j, b, i), for any $b \in [m]$. Moreover, z is adjacent to all vertices in $\{(i', b, i) : i' \neq j\}$, for all $b \in [m]$. By Proposition 3.2 (1), if $\mathcal{F} \cap U_{s,i}(k) \neq \varnothing$, then $\mathcal{F} \cap \left(U_{1,i}(k) \cup \ldots \cup U_{s-1,i}(k) \cup U_{s+1,i}(l) \cup \ldots \cup U_{m,i}(k)\right) = \varnothing$. Hence, we conclude that $|\mathcal{F}_i| \leq 1$. By a similar, argument, we also show that $|\mathcal{F}_i| \leq 1$, for any $i \in \{1, 2, \ldots, n\}$. Consequently, we have that

$$\alpha(\Gamma^k_{m,n}(k)) \leq \max\{k, n\}.$$

It is obvious that the two values in the upper bound are attained by $U_{1,1}(k)$ or by $\{(1, 1, c) : c \in [n]\}$. □

Corollary 3.5. *We have $\alpha\left(\Gamma^k_{m,n}(k)\right) = \max\left\{r, \frac{nr}{k}\right\}$.*

Proof. The result follows from the fact that $\alpha\left(\Gamma^k_{m,n}(k)\left[\overline{K_{\frac{r}{k}}}\right]\right) = \alpha(\Gamma^k_{m,n}(k))\alpha\left(\overline{K_{\frac{r}{k}}}\right)$. □

Now, we define another graph that is similar to $\Gamma^k_{m,n}(r)$ by introducing a set of permutations Σ. Let $\pi = \{P_1, P_2, \ldots, P_k\}$ be any uniform partition of $[r]$ into k parts. For any $b, b' \in [m]$ and $c, c' \in [n]$, we associate a permutation $\sigma^{c,c'}_{b,b'} \in \mathrm{Sym}(k)$ that depends only on b, b', c, c', and define the multiset of permutations

$$\Sigma = \left\{ \sigma_{b,b'}^{c,c'} : b, b' \in [m],\ c, c' \in [n] \right\}.$$

Now, we define the graph $\Gamma_{m,n}^{k,\Sigma}(r)$ to be the graph whose vertex set is

$$V = \{(a,b,c) : a \in [r], b \in [m], c \in [n]\}.$$

Two elements $(a,b,c), (a',b',c') \in V$ are adjacent if and only if

$$\begin{cases} c = c' \text{ and } b \neq b', \text{ or} \\ c \neq c' \text{ and } a \in P_i \text{ and } a' \in P_j \text{ such that } \sigma_{b,b'}^{c,c'}(i) \neq j. \end{cases}$$

If the $\sigma_{b,b'}^{c,c'}$ is the identity map for all $b, b' \in [m]$ and $c, c' \in [n]$, then $\Gamma_{m,n}^{k,\Sigma}(r) = \Gamma_{m,n}^{k}(r)$.

The graphs $\Gamma_{m,n}^{k}(r)$ and $\Gamma_{m,n}^{k,\Sigma}(r)$ are not necessarily isomorphic, but they are locally isomorphic in the sense that for all $b, b' \in [m]$ and $c, c' \in [n]$, the subgraphs induced by $\{(a,b,c) : a \in [r]\} \cup \{(a,b',c') : a \in [r]\}$ in $\Gamma_{m,n}^{k}(r)$ and $\Gamma_{m,n}^{k,\Sigma}(r)$ are isomorphic. A direct consequence of this is that the independence number of $\Gamma_{m,n}^{k}(r)$ and $\Gamma_{m,n}^{k,\Sigma}(r)$ are equal. In fact, we have

$$\Gamma_{m,n}^{k,\Sigma}(r) \cong \Gamma_{m,n}^{k,\Sigma'}(k)\left[\overline{K_{\frac{r}{k}}}\right]$$

for some multiset of permutations Σ' of $\mathrm{Sym}(k)$.

We omit the proof of the next lemma since it is similar to that of Lemma 3.4 and its corollary.

Lemma 3.6. *The independence number of $\Gamma_{m,n}^{k,\Sigma}(r)$ is equal to* $\max\{r, \frac{rn}{k}\}$.

We end this section by stating the No-Homomorphism Lemma. We recall that a homomorphism between a graph X and a graph Y is a map from the vertex set of X to the vertex set of Y which maps an edge to an edge.

Lemma 3.7 *(No-Homomorphism Lemma [2]). Let X be a graph and Y be a vertex-transitive graph. If there is a graph homomorphism from X to Y, then*

$$\frac{\alpha(Y)}{|V(Y)|} \leq \frac{\alpha(X)}{|V(X)|}.$$

4. Derangement graphs

Let H be a group and $C \subset H \setminus \{1\}$. Recall that the ***Cayley digraph*** $\mathrm{Cay}(H,C)$ is the digraph with vertex set equal to H, and for any $x, y \in H$, (x,y) is an arc if and only if $yx^{-1} \in C$. If $x^{-1} \in C$ whenever $x \in C$, then $\mathrm{Cay}(H,C)$ is a simple undirected graph. It is not hard to see that $\mathrm{Cay}(H,C)$ is vertex-transitive since its automorphism group contains a regular subgroup given by the right-regular representation of H.

Now, let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group. Recall that a ***derangement*** of G is a fixed-point-free permutation. A famous result of Jordan asserts that a finite transitive group of degree at least 2 always admits a derangement [15]. Let $\mathrm{Der}(G)$ be the set of all derangements of G. The ***derangement graph*** of G is the Cayley graph $\Gamma_G := \mathrm{Cay}(G, \mathrm{Der}(G))$.

The derangement graph is important in the analysis of the intersection density of the transitive group $G \leq \mathrm{Sym}(\Omega)$. Indeed, if $\mathcal{F} \subset G$ is intersecting, then for any $g, h \in \mathcal{F}$, hg^{-1} fixes an element of Ω. Hence, hg^{-1} is not in $\mathrm{Der}(G)$, and thus g and h are not adjacent in Γ_G. In other words, $\mathcal{F}$ is a coclique of Γ_G. Conversely, if $\mathcal{F}$ is a coclique of Γ_G, then any two permutations $g, h \in \mathcal{F}$ are such that $hg^{-1} \notin \mathrm{Der}(G)$, i.e., g and h agree on some element of Ω. Therefore, we conclude that $\mathcal{F} \subset G$ is intersecting if and only if $\mathcal{F}$ is a coclique in Γ_G. From this correspondence, we derive that

$$\rho(G) = \frac{\alpha(\Gamma_G)}{|G_\omega|}.$$

Since the derangement graph is a regular graph and vertex transitive, we can use various techniques to get an upper bound on the independence number of Γ_G. The next result will be useful for the results in this paper.

Lemma 4.1 *(Clique-coclique bound [11]). Let $X = (V,E)$ be a vertex-transitive graph. Then,*

$$\alpha(X)\omega(X) \leq |V|.$$

Moreover, if equality holds, then for any coclique of maximum size S and for any clique of maximum size T, we have $|S \cap T| = 1$.

We derive the following corollary from this lemma.

Corollary 4.2. *If $G \leq \mathrm{Sym}(\Omega)$ is transitive, then $\rho(G) \leq \frac{|\Omega|}{\omega(\Gamma_G)}$. In particular, if Γ_G has a clique of size $|\Omega|$, then $\rho(G) = 1$.*

Corollary 4.3. *If $G \leq \mathrm{Sym}(\Omega)$ is transitive of degree $3p$, where $p \geq 5$ is an odd prime, then $\rho(G) \leq 3$.*

Proof. In [18] Marušič showed that any transitive group of degree mq, where q is a prime and $m \leq q$, admits a semiregular element of order q. Let $g \in G$ be a semiregular element of order p. The subgroup $\langle g \rangle$ is a clique of size p in the derangement graph Γ_G. By Corollary 4.2, we have $\rho(G) \leq \frac{3p}{p} = 3$. $\square$

5. Analysis of the intersection density of groups of degree $3p$

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive and $|\Omega| = 3p$, where $p \geq 5$ is an odd prime. We recall the following lemma.

Lemma 5.1 *([13]). Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group. If $\mathcal{B}$ is a G-invariant partition of Ω and $H \leq G$ is a semiregular subgroup whose orbits-partition is equal to $\mathcal{B}$, then $\rho(G) \leq \rho(\overline{G})$.*

If G is primitive, then $\rho(G) = 1$ by [22], so we may assume that G is imprimitive. As G is imprimitive of degree $3p$, it admits only blocks of size 3 or p. If G admits a block of size p from a G-invariant partition $\mathcal{B}$ of Ω, then we can also show that $\rho(G) = 1$. Indeed, Marušič showed in [18] that a transitive group of degree mp, for $m \leq p$, admits a semiregular element of order p. If H is the semiregular subgroup obtained from such a semiregular element, it is straightforward to see that the set of orbits of H must be equal to $\mathcal{B}$. By Lemma 5.1, we conclude that $\rho(G) \leq \rho(\overline{G}) = 1$, since $\overline{G}$ is transitive of prime degree.

If G admits at least two non-trivial G-invariant partitions, then $\rho(G) = 1$ by a result in [22, Section 3]. Therefore, we may assume that G admits a unique non-trivial G-invariant partition,

$$\mathcal{B} = \{B_1, B_2, \ldots, B_p\} \tag{1}$$

where the blocks are $B_i = \{x_i, y_i, z_i\}$, for any $1 \leq i \leq p$. We may distinguish two cases from hereon in our analysis.

5.1. The quasiprimitive case

If $G \leq \mathrm{Sym}(\Omega)$ is quasiprimitive (i.e., all of its non-trivial normal subgroups are transitive), then $\ker(G \to \overline{G})$ is trivial, since it cannot be transitive on Ω. Conversely, assume that $\ker(G \to \overline{G})$ is trivial. For any non-trivial normal subgroup $N \trianglelefteq G$, the orbits of N form a G-invariant partition of Ω [6, Theorem 1.6A] and so the orbits-partition of N is either trivial or equal to $\mathcal{B}$. Since $N \neq 1$, the orbits-partition of N cannot be $\{\{\omega\} : \omega \in \Omega\}$. Similarly, the orbits-partition of N cannot be equal to $\mathcal{B}$, otherwise we would have that $N \leq \ker(G \to \overline{G})$ but $N \neq 1$ and $\ker(G \to \overline{G})$ is trivial. Therefore, the orbits-partition of N is equal to $\{\Omega\}$, and so N is transitive.

Thus, we have proved that $\ker(G \to \overline{G})$ is trivial if and only if G is quasiprimitive. In this case, $G \cong \overline{G}$, and G admits two faithful actions of degree $3p$ and p. Let $\omega \in \Omega$ and assume that $\omega \in B$, for some $B \in \mathcal{B}$. Let $G_{\{B\}}$ be the setwise stabilizer of the set B in G. Note that $G_{\{B\}} = \overline{G}_B$ and if $g \in G_\omega$, then $g \in G_{\{B\}}$. Therefore, we conclude that $G_\omega \leq G_{\{B\}}$. Hence, G admits two actions, which are permutation equivalent to the actions of G on cosets of $G_{\{B\}}$ (primitive of degree p) and on cosets of G_ω (imprimitive and quasiprimitive of degree $3p$).

Using the classification of transitive groups of prime degree, it is not hard to show (see [3]) that G is almost simple, and

$$\mathrm{PSL}_n(q) \leq G \leq \mathrm{P\Gamma L}_n(q)$$

where $n \geq 1$ is an integer, $p = \frac{q^n-1}{q-1}$, and the point-stabilizer of $\mathrm{PSL}_n(q)$ in its action on the projective space $\mathrm{PG}_{n-1}(q)$ admits a subgroup of index 3.

5.2. The genuinely imprimitive case

Assume that $L \trianglelefteq G$ is non-trivial and intransitive. As $L \trianglelefteq G$, its orbits-partition is a G-invariant partition of G, and since it is non-trivial and intransitive, its orbits-partition is equal to $\mathcal{B}$. Consequently, $L \leq \ker(G \to \overline{G})$.

If $N \trianglelefteq G$ is a minimal normal subgroup of G, then either N is transitive or it is intransitive and thus contained in $\ker(G \to \overline{G})$. As G is not quasiprimitive, note that there is always a minimal normal subgroup of G contained in $\ker(G \to \overline{G})$.

Let $N \trianglelefteq G$ be a minimal normal subgroup of G such that $N \leq \ker(G \to \overline{G})$. By [6, Theorem 4.3A], $N = T_1 \times T_2 \times \ldots \times T_k$, where $k \geq 1$ is a positive integer, $(T_i)_{i=1,\ldots,m}$ are simple normal subgroups of N and are conjugate in G. We may distinguish the cases where one of the factors (and therefore all, by conjugation) is abelian or not.

- If T_1 is non-abelian (and therefore all other T_i, for $1 \leq i \leq k$), then it was proved in [3] that $\rho(G) = 1$.

- If T_1 is abelian, then N is an elementary abelian 3-group [3]. It was also proved in [3] that if $\ker(G \to \overline{G})$ contains a derangement, then $\rho(G) = 1$. Consequently, the only transitive groups $G \leq \mathrm{Sym}(\Omega)$ of interest in this case are those with the property that $\ker(G \to \overline{G})$ is derangement-free.

Consequently, we make the following assumption for the remainder of the paper.

Assumption 1. Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group of degree $3p$ admitting the G-invariant partition defined in (1) as its only non-trivial G-invariant partition. Let $K := \ker(G \to \overline{G})$. Assume that $K \neq 1$ is derangement-free and that any minimal normal subgroup of G contained in K is an elementary abelian 3-group.

In the next sections, we will analyze the possible cases for the intersection density under Assumption 1.

6. The solvable case

Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group satisfying Assumption 1. As $\overline{G} \leq \mathrm{Sym}(\mathcal{B})$ is transitive of degree p, it is either 2-transitive or solvable, and thus a subgroup of $\mathrm{AGL}_1(p)$. We assume further that $\overline{G}$ is solvable.

Under these assumptions, we have that $\overline{G} = \langle \alpha \rangle \rtimes \langle \beta \rangle$, where $o(\alpha) = p$ and $o(\beta) = d \mid (p-1)$. Moreover, β fixes B_1 and acts as a product of $\frac{p-1}{d}$ many d-cycles on $\mathcal{B} \setminus \{B_1\}$. In other words, $\overline{G}$ is a Frobenius group.

Let $a \in G$ such that $\overline{a} = \alpha$. A result of Marušič in [18, Theorem 3.4] shows that the group G of degree $3p$ always has a semiregular element of order p. That is, we may assume that a is a product of 3 cycles of length p. In other words, $o(a) = o(\alpha) = p$. Therefore,

$$\langle K, a \rangle = K\langle a \rangle = K \rtimes \langle a \rangle$$

is a transitive group where K is intersecting, since it is derangement-free [20]. As the intersection density of a transitive subgroup of G is at most 3 by Corollary 4.3, it is not hard to see that the intersecting density of $K \rtimes \langle a \rangle$ is exactly equal to 3. Indeed, K is an intersecting set of size $\frac{|K \rtimes \langle a \rangle|}{p}$.

Next, let $b \in G$ such that $\overline{b} = \beta$. Since $\overline{b} = \beta$, then $d = o(\beta) \mid o(b)$, so there exists a positive integer $r \geq 1$ such that $o(b) = dr$. Also, since $\overline{G} = \langle \alpha \rangle \rtimes \langle \beta \rangle$, there exists $t \in \{1, \ldots, p-1\}$ such that $\beta\alpha\beta^{-1} = \alpha^t$. Consequently, we have $\overline{bab^{-1}} = \overline{a^t}$, so there exists $h \in K$ such that

$$bab^{-1} = ha^t. \tag{2}$$

Define the group

$$G(a, b) := \langle K, a, b \rangle. \tag{3}$$

Since $\overline{G} = \langle \alpha \rangle \rtimes \langle \beta \rangle$, for any $g \in G$, there exists $i \in \{0, 1, \ldots, p-1\}$ and $j \in \{0, 1, \ldots, d-1\}$ such that $\overline{g} = \alpha^i\beta^j = \overline{a^ib^j}$. Therefore, there exists $k \in K$ such that $g = ka^ib^j \in \langle K, a, b \rangle = G(a, b)$. In other words, $G = G(a, b)$.

Proposition 6.1. *If $o(b) = rd$, then $\langle K, a \rangle \cap \langle b \rangle = \langle b^d \rangle$. In particular, if $o(b) = d$, then $G(a, b) = (K \rtimes \langle a \rangle) \rtimes \langle b \rangle$.*

Proof. Let $b^j \in \langle K, a \rangle \cap \langle b \rangle$. It is easy to see that elements of the form $ka^i \in \langle K, a \rangle = K \rtimes \langle a \rangle$, where $i \neq 0$, must be derangements, so they cannot be in $\langle b \rangle$. Therefore, an element of the intersection $\langle K, a \rangle \cap \langle b \rangle$ must be in K. We have $b^j \in K$ if and only if $\beta^j = \overline{1}$, which can only happen when $d \mid j$. Thus, $b^j \in \langle b^d \rangle$. The converse follows immediately from that fact that $b^d \in K$.

If $o(b) = d$, then the second part of the proposition is trivial. □

The following result is an immediate consequence of the previous proposition.

Corollary 6.2. $|G(a, b)| = |K|pd$.

Lemma 6.3. *An element of K has order dividing* 6.

Proof. Let $g \in K$. Since K fixes each element of $\mathcal{B}$ setwise, the restriction of $g \in K$ onto a block in $\mathcal{B}$ has order 1, 2, or 3. Therefore, $o(g) \mid 6$. □

Since $B_1^b = B_1$, the restriction of $b_{|B_1} = \sigma \in \mathrm{Sym}(B_1)$. Hence, σ either has a fixed point or it is a 3-cycle. If σ is a 3-cycle, then we can find an element $h \in K$ such that $h_{|B_1} = \sigma^{-1}$, and so we can replace b with hb instead. Thus we may assume, without loss of generality, that b fixes an element in $B_1 \in \mathcal{B}$. From this, b either fixes B_1 pointwise or we may assume that its restriction on B_1 is the permutation $(x_1\ y_1)$. As we will see in the next section, it is imperative to know whether $b_{|B_1}$ is trivial or a transposition.

Remark 6.4. If K admits an involution, then we may conjugate this involution so that the resulting permutation does not fix all points in B_1. By conjugating with an appropriate element of order 3 in the minimal normal subgroup N, we obtain an involution σ whose restriction onto B_1 is $(x_1\ y_1)$. As $\sigma \in K$, we may replace b with σb, as this element fixes B_1 pointwise.

In the following two sections, we will consider two cases, depending on whether K has an involution or no.

7. The kernel K has no involutions

Let $G \leq \mathrm{Sym}(\Omega)$ be a group satisfying Assumption 1 and assume that K has no involutions. Since K does not have an involution, every element of K has order 3, i.e., it is an elementary abelian 3-group. Assume that $\overline{G} = \langle \alpha \rangle \rtimes \langle \beta \rangle \leq \mathrm{AGL}_1(p)$ is non-cyclic and $\beta\alpha\beta^{-1} = \alpha^t$, for some $t \in \mathbb{Z}$ such that $\gcd(t, p) = 1$. Let a be a semiregular element such that $\overline{a} = \alpha$ and $b \in G$ such that $\overline{b} = \beta$ with $o(\beta) = d$ and $o(b) = rd$. Since $\beta\alpha\beta^{-1} = \alpha^t$, there exists $h \in K$ such that $bab^{-1} = ha^t$. Henceforth, let $G(a, b)$ be the group defined in (3).

Assume without loss of generality that

$$a = \left(x_1\ x_2\ \ldots\ x_p\right)(y_1\ y_2\ \ldots\ y_p)(z_1\ z_2\ \ldots\ z_p). \tag{4}$$

Consequently, $\alpha = \overline{a} = (B_1\ \ B_2\ \ \ldots\ \ B_p)$. As $\beta\alpha\beta^{-1} = \alpha^t$, we know that $t^d \equiv 1 \pmod p$. Hence, we must have for any $i \in \{2, 3, \ldots, p\}$ that

$$B_i^\beta = B_{1+(i-1)t}.$$

Therefore, the cycle of β containing B_i must be of the form

$$\left(B_i\ B_{1+(i-1)t}\ B_{1+(i-1)t^2}\ \ldots B_{1+(i-1)t^{d-1}}\right).$$

Lemma 7.1. *For any $u, v \in \mathbb{Z}$ with $d \nmid v$ (or equivalently, $\overline{b^v} \neq \overline{1}$), the element a^ub^v is conjugate to $k_{u,v}b^v$ in $G(a, b)$, for some $k_{u,v} \in K$.*

Proof. To show that a^ub^v is conjugate to an element in Kb^v, it is enough to show that $\overline{a^ub^v} = \alpha^u\beta^v$ and $\overline{Kb^v} = \beta^v$ are conjugate in $\overline{G}$. The elements $\alpha^u\beta^v$ and β^v are conjugate if and only if there exists $g \in G$ such that $\alpha^u\beta^v = \overline{g}\beta^v\overline{g^{-1}}$.

Note that since $\beta\alpha\beta^{-1} = \alpha^t$ and d is the order of β, we must have that d is the smallest positive integer such that $t^d - 1$ is divisible by p, and $p \mid (t^s - 1)$ if and only if $d \mid s$. Therefore, we know that $\gcd(t^v - 1, p) = 1$, as $d \nmid v$. If $n \in \{0, 1, \ldots, p-1\}$ is the unique solution of $(1 - t^v)n \equiv u \pmod p$, then we have

$$\alpha^n\beta^v\alpha^{-n} = \alpha^{n-nt^v}\beta^v = \alpha^{n(1-t^v)}\beta^v = \alpha^u\beta^v.$$

Consequently, $\alpha^u\beta^v = \overline{a^ub^v}$ and $\beta^v = \overline{b^v}$ are conjugate, and so there exists $k_{u,v} \in K$ such that a^ub^v is conjugate to $k_{u,v}b^v$. This completes the proof. $\square$

Remark 7.2. Assume that a^ub^v leaves B_i invariant, for some $i \in \{1, 2, \ldots, p\}$. Since

$$B_i^{a^ub^v} = B_{i+u}^{b^v} = B_{1+(i+u-1)t^v},$$

the unique element fixed by a^ub^v setwise is B_i, where i is the unique solution modulo p to the modular equation

$$(t^v - 1)i \equiv (1 - u)t^v - 1 \pmod p.$$

The structure of the derangement graph of $G(a, b)$ depends on the number of fixed points of the restriction $b_{|B_1}$ of b onto B_1. We distinguish the cases whether b fixes 1 or 3 points of B_1.

7.1. Case 1. b fixes B_1 pointwise

Throughout this subsection, we make the assumption that b fixes B_1 pointwise.

Now, we are ready to prove the first main theorem of this section.

Theorem 7.3. *Consider the group $G = G(a, b)$ defined in (3) satisfying Assumption 1, where a is a semiregular element of order p, and b is an element of order rd. Assume that K does not have an involution. If b fixes B_1 pointwise, then*

$$\rho(G(a, b)) = \max\{1, \tfrac{3}{d}\}.$$

In particular, if G admits an element b of order at least 3 fixing B_1 pointwise and $\overline{b} = \beta$, then $\rho(G) = 1$.

Proof. Let $\Gamma_{G(a,b)}$ be the derangement graph of $G(a,b)$. Assume that $|K|=m$. By Assumption 1, K is derangement-free, so it is a coclique of the derangement graph $\Gamma_{G(a,b)}$. Recall that a right-transversal is a system of distinct representatives of right cosets of K. The set $\{a^ub^v : u \in \{0,1,\ldots,p-1\} \text{ and } v \in \{0,1,\ldots,rd-1\}\}$ is not a right-transversal of K in $G(a,b)$ since $b^d \in K$. It is not hard to see however that $\{a^ub^v : u \in \{0,1,\ldots,p-1\} \text{ and } v \in \{0,1,\ldots,d-1\}\}$ is a right-transversal of K in $G(a,b)$.

Claim 1. *If $v=v'$ and $u \neq u'$, then the subgraph of $\Gamma_{G(a,b)}$ induced by Ka^ub^v and $Ka^{u'}b^v$ is a complete bipartite graph $K_{m,m}$.*

Proof of Claim 1. Let $k,k' \in K$. Then, $ka^ub^v(k'a^{u'}b^v)^{-1} = ka^{u-u'}\left(k'\right)^{-1} = k''a^{u-u'}$, for some $k'' \in K$. As $u \neq u'$, it follows that $k''a^{u-u'}$ is a derangement. Hence, the subgraph of $\Gamma_{G(a,b)}$ induced by $Ka^ub^v \cup Ka^{u'}b^v$ is a complete bipartite graph. □

Claim 2. *If $v \neq v'$, then the subgraph of $\Gamma_{G(a,b)}$ induced by $Ka^ub^v \cup Ka^{u'}b^{v'}$ is the lexicographic product $X[\overline{K_{\frac{m}{3}}}]$, where X is the complete bipartite graph $K_{3,3}$ with a perfect matching removed (see Fig. 2).*

Proof of Claim 2. Let $B_i \in \mathcal{B}$ be the block fixed by

$$\alpha^{u'}\beta^{v'}(\alpha^u\beta^v)^{-1} = \alpha^{u'}\beta^{v'-v}\alpha^{-u} = \alpha^{u'-ut^{v'-v}}\beta^{v'-v}.$$

Let $K_{x_i} = \{k \in K : \ x_i^k = x_i\}$ and let $c \in K$ such that the restriction of c onto B_i is $c_{|B_i} = (x_i \ \ y_i \ \ z_i)$. Clearly, $\langle c\rangle$ is a right-transversal of K_{x_i} in K. Therefore, $K = K_{x_i} \cup K_{x_i}c \cup K_{x_i}c^2$ is a disjoint union.

Now consider two arbitrary elements $kc^\ell a^ub^v \in Ka^ub^v$ and $k'c^{\ell'}a^{u'}b^{v'} \in Ka^{u'}b^{v'}$, for some $\ell,\ell' \in \{0,1,2\}$ and $k,k' \in K_{x_i}$. We claim that if these two elements intersect on $\omega \in \Omega$, then $\omega \in B_i$. Indeed, if we have $\omega \in \Omega$ such that

$$\omega^{kc^\ell a^ub^v} = \omega^{k'c^{\ell'}a^{u'}b^{v'}},$$

then

$$\omega^{kc^\ell} = \omega^{k'c^{\ell'}a^{u'}b^{v'}(a^ub^v)^{-1}}.$$

As $kc^\ell, k'c^{\ell'} \in K$ and B_i is the unique block fixed by $a^{u'}b^{v'}(a^ub^v)^{-1}$ setwise, we conclude that an element $\omega \in \Omega$ on which kc^ℓ and $k'c^{\ell'}a^{u'}b^{v'}(a^ub^v)^{-1}$ agree must be in B_i.

Now, we note that

$$a^{u'}b^{v'}(a^ub^v)^{-1} = za^{u'-ut^{(v'-v)}}b^{(v'-v)}$$

for some $z \in K$. By Lemma 7.1, $a^{u'-ut^{(v'-v)}}b^{(v'-v)}$ is conjugate to $k''b^{(v'-v)}$, for some $k'' = k_{u'-ut^{(v'-v)},v'-v} \in K$. If $k''_{|B_1}$ is trivial, then the permutation $\left(a^{u'-ut^{(v'-v)}}b^{(v'-v)}\right)_{|B_i}$ is trivial, whereas $\left(a^{u'-ut^{(v'-v)}}b^{(v'-v)}\right)_{|B_i}$ is a 3-cycle if $k''_{|B_1}$ has order 3. Therefore, $\left(a^{u'}b^{v'}(a^ub^v)^{-1}\right)_{|B_i}$ is of order 1 or 3, since $z \in K$.

Note that $k,k' \in K_{x_i}$ fix B_i pointwise. Hence, kc^ℓ and $k'c^{\ell'}a^{u'}b^{v'}(a^ub^v)^{-1}$ are intersecting if and only if c^ℓ and $c^{\ell'}a^{u'}b^{v'}(a^ub^v)^{-1}$ are intersecting on an element of B_i. Since $a^{u'}b^{v'}(a^ub^v)^{-1}_{|B_i}$ is of order 1 or 3, we have that c^ℓ and $c^{\ell'}a^{u'}b^{v'}(a^ub^v)^{-1}$ are intersecting on B_i for a unique $s \in \{0,1,2\}$ such that $\ell-\ell' = s$.

Since there are only three choices for ℓ and ℓ', we conclude overall that the subgraph of $\Gamma_{G(a,b)}$ induced by Ka^ub^v and $Ka^{u'}b^{v'}$ is equal to the lexicographic product $X[\overline{K_{\frac{m}{3}}}]$, where X is the graph in Fig. 2. This completes the proof of Claim 2. □

Hence, the derangement graph of $G(a,b)$ is isomorphic to $\Gamma_{p,d}^{3,\Sigma}(m)$, for some multiset of permutations Σ of Sym(3). We conclude that

$$\rho(G(a,b)) = \max\left\{\frac{3p|K|}{|G(a,b)|}, \frac{3pd|K|}{3|G(a,b)|}\right\} = \max\left\{\frac{3}{d},1\right\}. \quad \square$$

7.2. *Case 2. b has one fixed point on B_1*

We will show that whenever $b_{|B_1}$ is a transposition, then the situation is quite different to the previous subsection. We first note that $o(b) = rd$ is even since $b_{|B_1}$ is a transposition.

First, assume that d is odd. Then, b^2 fixes B_1 pointwise and $\overline{b^2} = \beta^2$. Hence, $o\left(\beta^2\right) = \frac{d}{\gcd(d,2)} = o(\beta)$. From this, we deduce that $\langle\beta^2\rangle = \langle\beta\rangle$, which in turn implies that $b \in K\langle b^2\rangle$. Further, one can easily deduce that $G(a,b) = G(a,b^2)$. As b^2 fixes B_1 pointwise, we can use Theorem 7.3 to show that

$$\rho(G(a,b)) = \rho(G(a,b^2)) = \max\{1, \tfrac{3}{d}\}.$$

We therefore assume that d is even for the remainder of this section.

We derive an important corollary from Lemma 7.1 by noting that $b_{|B_1}$ is an involution and K is an elementary abelian 3-group. By Lemma 7.1, $a^u b^v$ fixes the same number of points as $k_{u,v}b^v$ (see the statement of Lemma 7.1). If v is even, then $\left(k_{u,v}b^v\right)_{|B_1} = \left(k_{u,v}\right)_{|B_1}$. If v is odd, then by the fact that $b_{|B_1}$ is an involution, we know that $(k_{u,v}b^v)_{|B_1} = \left(k_{u,v}b\right)_{|B_1}$. Since K is an elementary abelian 3-group, the element $k_{u,v}b_{|B_1}$ is an involution, and thus fixes one point of B_1.

Corollary 7.4. *Let $u \in \{0,1,\ldots,p-1\}$ and $v \in \{1,2,\ldots,d-1\}$. Assume that B_i is fixed by $\alpha^u\beta^v$. If v is even, then a^ub^v fixes B_i pointwise or fixes no points in B_i. If v is odd, then a^ub^v fixes a unique element from B_i.*

Theorem 7.5. *Consider the group $G = G(a,b)$ defined in (3) satisfying Assumption 1, where a is a semiregular element of order p, and b is an element of order rd. Assume that K does not have an involution. If $b_{|B_1}$ is a transposition and $d = o(\beta)$ is even, then*

$$\rho(G(a,b)) = \max\{1, \tfrac{6}{d}\}.$$

In particular, if $d \le 6$, then $\rho(G(a,b)) \in \{1, \tfrac{3}{2}, 3\}$.

Proof. Let $\Gamma_{G(a,b)}$ be the derangement graph of $G(a,b)$. Assume that $|K| = m$. By Assumption 1, K is derangement-free, so it is a coclique of the derangement graph $\Gamma_{G(a,b)}$. Consider the right-transversal of K in $G(a,b)$ given by $\{a^ub^v : u \in \{0,1,\ldots,p-1\} \text{ and } v \in \{0,1,\ldots,d-1\}\}$.

The proof of the following claim is omitted since it is similar to its analogue in Theorem 7.3.

Claim 3. *If $v = v'$ and $u \ne u'$, then the subgraph of $\Gamma_{G(a,b)}$ induced by $Ka^ub^v \cup Ka^{u'}b^v$ is a complete bipartite graph $K_{m,m}$.*

Claim 4. *If $v \ne v'$, then the subgraph of $\Gamma_{G(a,b)}$ induced by Ka^ub^v and $Ka^{u'}b^{v'}$ is empty if $v'-v$ is odd, and equal to $X[\overline{K_{\frac{m}{3}}}]$ where X is the graph in Fig. 2 if $v'-v$ is even.*

Proof of Claim 4. Let B_i be the unique block fixed by $\alpha^{u'}\beta^{v'}(\alpha^u\beta^v)^{-1}$. Let $c \in K$ such that $\langle c\rangle$ is a right-transversal of K_{x_i} in K. Similar to the proof of Claim 2 in Theorem 7.3, we examine the edges between $K_{x_i}c^\ell a^ub^v$ and $K_{x_i}c^{\ell'}a^{u'}b^{v'}$, for all $\ell,\ell' \in \{0,1,2\}$. For any $k,k' \in K_{x_i}$, we have that $kc^\ell a^ub^v$ and $k'c^{\ell'}a^{u'}b^{v'}$ are intersecting if and only if kc^ℓ and $k'c^{\ell'}a^{u'}b^{v'}\left(a^ub^v\right)^{-1}$ are intersecting on B_i. Here, it is worthwhile to note that

$$a^{u'}b^{v'}\left(a^ub^v\right)^{-1} = za^{u'-ut^{(v'-v)}}b^{(v'-v)},$$

for some $z \in K$.

- If $v-v'$ is even, then $b^{v'-v}$ fixes B_1 pointwise, and by Corollary 7.4 we know that the permutation $\left(za^{u'-ut^{(v'-v)}} \times b^{(v'-v)}\right)_{|B_i}$ has the same number of fixed points on B_i as
$$\left(zk_{u'-ut^{(v'-v)},v'-v}\right)_{|B_1}.$$
The latter has 1 or 3 fixed points. Using the same argument as Claim 2 in the proof of Theorem 7.3, we conclude that the subgraph induced by Ka^ub^v and $Ka^{u'}b^{v'}$ is isomorphic to $X[\overline{K_{\frac{m}{3}}}]$, where X is the graph given in Fig. 2.
- If $v'-v$ is odd, then by Corollary 7.4, $a^{u'-ut^{(v'-v)}}b^{(v'-v)}$ fixes a unique point of B_i, and so does $za^{u'-ut^{(v'-v)}}b^{(v'-v)}$ since $z \in K$ has order 1 or 3. Let $c \in K$ be such that $\langle c\rangle$ is a right-transversal of K_{x_i} in K. Let $\ell,\ell' \in \{0,1,2\}$. For any $n,n' \in K_{x_i}$, the elements $nc^\ell a^ub^v$ and $n'c^{\ell'}a^{u'}b^{v'}$ always agree on an element of B_i since $n,n' \in K_{x_i}$ fix B_i pointwise, and the permutations $c^\ell_{|B_i}$ and $\left(c^{\ell'}a^{u'}b^{v'}\left(a^ub^v\right)^{-1}\right)_{|B_i} = \left(c^{\ell'}za^{u'-ut^{(v'-v)}}b^{v'-v}\right)_{|B_i}$ are respectively elements of order belonging to $\{1,3\}$ and an involution. Consequently, there is no edge between Ka^ub^v and $Ka^{u'}b^{v'}$ in $\Gamma_{G(a,b)}$. $\square$

By combining Claim 3 and Claim 4, it is not hard to show that $\Gamma_{G(a,b)}$ is the union of two graphs isomorphic to $\Gamma^{3,\Sigma}_{p,t_1}(|K|)$ and $\Gamma^{3,\Sigma}_{p,t_2}(|K|)$, where

$$t_1 = |\{0 \le i \le d-1 : i \text{ is odd }\}| = \tfrac{d}{2} \text{ and } t_2 = |\{0 \le i \le d-1 : i \text{ is even }\}| = \tfrac{d}{2}.$$

The independence number of the graphs $\Gamma^{3,\Sigma}_{p,t_1}(|K|)$ and $\Gamma^{3,\Sigma}_{p,t_2}(|K|)$ are both equal to $\max\{t_1|K_{x_1}|, |K|\} = \max\{t_2|K_{x_1}|, |K|\}$. The independence number of the union of the two graphs is

$\max\{|K_{x_1}|d, 2|K|\}$.

Therefore,

$$\rho(G(a,b)) = \max\left\{1, \max\left\{\frac{6}{d}, 1\right\}\right\} = \max\left\{1, \frac{6}{d}\right\}.$$

If $\frac{d}{2} \geq 3$, then $\rho(G(a,b)) = 1$. If $\frac{d}{2} < 3$, then using the fact that d is even we have $d \in \{2,4\}$ and $\rho(G(a,b)) \in \{\frac{3}{2}, 3\}$. This completes the proof. □

8. *K* has an involution

Throughout this section, we let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group satisfying Assumption 1 and we assume that K admits an involution. Assume that $\overline{G} = \langle \alpha \rangle \rtimes \langle \beta \rangle$, and recall that a and b are two elements of G such that $\overline{a} = \alpha$ and $\overline{b} = \beta$. Recall that $o(\alpha) = o(a) = p$, $o(\beta) = d$, and $o(b) = rd$, for some positive integer r. Since K has an involution, we may assume that b fixes B_1 pointwise (see Remark 6.4). As we have seen previously, we have $G = G(a,b)$.

For any distinct $g, g' \in K$ of order 3, we have $gg'g^{-1} = g'$, and thus $\langle g, g' \rangle = C_3 \times C_3$ unless $g' = g^{-1}$. Let E be the subgroup generated by all elements of order 3 in K. By commutativity of the elements of order 3 in K, we know that E is an elementary abelian 3-group.

Let $|K| = m$. The set $\{a^u b^v : \ u \in \{0,1,\dots,p-1\}, v \in \{0,1,\dots,d-1\}\}$ is again a right-transversal of K in $G(a,b)$. For any $u, u' \in \{0,1,\dots,d-1\}$ and $v, v' \in \{0,1,\dots,d-1\}$, we have

$$\left(Ka^u b^v\right)\left(Ka^{u'} b^{v'}\right) = Ka^{u' - ut^{v'-v}} b^{v'-v}.$$

Remark 8.1.

(a) For any $u \in \{0,1,\dots,p-1\}$ and $v \in \{1,\dots,d-1\}$, the element $a^u b^v$ fixes a certain block B_i setwise. Since K admits an involution, we may assume that $a^u b^v$ fixes this block pointwise. To see this, consider the permutation $\sigma = \left(a^u b^v\right)_{B_i}$. If σ is the identity, then the statement holds. If σ is a transposition, then there exists an element $g \in K$ such that $g_{|B_i} = \sigma$. So $(ga^u b^v)_{B_i}$ is the identity permutation, and we may replace $a^u b^v$ with $ga^u b^v$. Finally, if σ is a 3-cycle, then there exists $g \in K$ such that $g_{|B_i} = \sigma^{-1}$, so $(ga^u b^v)_{B_i}$ is the identity permutation, and again we may replace $a^u b^v$ with $ga^u b^v$.

(b) In contrast to (a), the case where K has no involution is quite different. Indeed, if $\left(a^u b^v\right)_{|B_i}$ is an involution, then one cannot multiply it with an element of K to make the resulting permutation fix B_i pointwise.

Now, we proceed with the proof. Let $\{k_1 = 1, k_2, \dots, k_s\}$ be a right-transversal of E in K, where s is the index of E in K. Then,

$$K = \bigcup_{j=1}^{s} Ek_j$$

Also, let $c \in E$ such that $c_{|B_i} \neq 1$. Then,

$$E = \bigcup_{\ell=0}^{2} E_{x_i} c^{\ell}.$$

Hence,

$$K = \bigcup_{j=1}^{s} \bigcup_{\ell=0}^{2} E_{x_i} c^{\ell} k_j.$$

In the next lemma, we show that every non-trivial element of the right-transversal $\{k_1 = 1, k_2, \dots, k_s\}$ can be assumed to have order 2.

Lemma 8.2. *There exists a right-transversal of the subgroup E of K consisting of the identity and involutions.*

Proof. Recall that $\{k_1 = 1, k_2, \dots, k_s\}$ is a right-transversal. Since E contains all elements of order 3 of K, by Lemma 6.3, we have $o(k_i) \in \{2,6\}$ for $i \neq 1$. If $o(k_i) = 2$, then we are done. If $o(k_i) = 6$, then we know that $k_i = k_i^{-2} k_i^3$, and that $o(k_i^{-2}) = 3$ and $o(k_i^3) = 2$. We conclude that $k_i \in Ek_i^3$, so $Ek_i = Ek_i^3$. In other words, there always exists a right-transversal consisting of the identity and involutions. □

From the above lemma, we assume henceforth $\{1, k_2, \ldots, k_s\}$ is such that $o(k_2) = \ldots = o(k_s) = 2$.

Theorem 8.3. *Consider the group $G = G(a, b)$ defined in (3) satisfying Assumption 1, where a is a semiregular element of order p, and b is an element of order rd. If K has an involution, then*

$$\rho(G(a,b)) = \max\left\{1, \frac{3}{d}\right\}.$$

Proof. Let $u, u' \in \{0, 1, 2, \ldots, p-1\}$ and $v, v' \in \{0, 1, \ldots, d-1\}$, and let us determine the edges induced by the vertices in $Ka^u b^v \cup Ka^{u'} b^{v'}$ in $\Gamma_{G(a,b)}$.

The proof of the following claim is omitted since it is similar to the proof of Claim 1.

Claim 5. *If $v = v'$ and $u \neq u'$, then the subgraph of $\Gamma_{G(a,b)}$ induced by $Ka^u b^v \cup Ka^{u'} b^{v'}$ is a complete bipartite $K_{m,m}$, where $m = |K|$.*

Suppose that B_i is the unique block fixed setwise by $a^{u'} b^{v'} \left(a^u b^v\right)^{-1}$.

Claim 6. *If $v \neq v'$, then the subgraph of $\Gamma_{G(a,b)}$ induced by $Ka^u b^v \cup Ka^{u'} b^{v'}$ contains a subgraph isomorphic to the disjoint union of $s = [K : E]$ copies of $X[\overline{K_{\frac{|E|}{3}}}]$, where X is the graph in Fig. 2.*

Proof of Claim 6. Recall that B_i is the unique element of $\mathcal{B}$ fixed by $(a^u b^v)(a^{u'} b^{v'})^{-1}$ setwise. By Remark 8.1, we may assume that $(a^u b^v)(a^{u'} b^{v'})^{-1}$ fixes B_i pointwise. We note that a vertex in $Ka^u b^v$ intersects a vertex in $Ka^{u'} b^{v'}$ on $\omega \in \Omega$ only if $\omega \in B_i$.

Fix $j \in \{1, 2, \ldots, s\}$. We will show now that for any $\ell \in \{0, 1, 2\}$, there exists a unique $\ell' \in \{0, 1, 2\}$ such that the subgraph of $\Gamma_{G(a,b)}$ induced by $Ec^\ell k_j \cup Ec^{\ell'} k_j$ is isomorphic to $X[\overline{K_{\frac{|E|}{3}}}]$. Since the elements of E_{x_i} fix B_i pointwise, we first note that one only needs to determine whether $c^\ell k_j$ and $c^{\ell'} k_j (a^u b^v)(a^{u'} b^{v'})^{-1}$ are adjacent, or equivalently do not fix a point in B_i. As $(a^u b^v)(a^{u'} b^{v'})^{-1}$ also fixes B_i pointwise, we only need to check the adjacency between $c^\ell k_j$ and $c^{\ell'} k_j$. We note that the restrictions of these two elements onto B_i are permutation of $\mathrm{Sym}(B_i)$ that have the same cycle type. Hence, for any $\ell \in \{0, 1, 2\}$ there exists a unique $s \in \{0, 1, 2\}$ such that $\ell - \ell' = s$ and for which vertices in $E_{x_i} c^\ell k_j a^u b^v \cup E_{x_i} c^{\ell'} k_j a^{u'} b^{v'}$ form a coclique. For any other values of $\ell - \ell'$, the vertices in $E_{x_i} c^\ell k_j a^u b^v \cup E_{x_i} c^{\ell'} k_j a^{u'} b^{v'}$ induce a complete bipartite graph.

Consequently, the subgraph of $\Gamma_{G(a,b)}$ induced by $Ek_j a^u b^v \cup Ek_j a^{u'} b^{v'}$ is isomorphic to $X[\overline{K_{\frac{|E|}{3}}}]$. The s copies are obtained by varying $j \in \{1, 2, \ldots, s\}$. This completes the proof of Claim 6. □

Claim 7. *There is a homomorphism from $\Gamma^{\Sigma}_{p,d}(|E|)$ to $\Gamma_{G(a,b)}$, for some multiset Σ of permutations of* Sym(3).

Proof of Claim 7. Using Claim 6, it is easy to see that there is in fact a multiset of permutation Σ of $\mathrm{Sym}(3)$ for which the graph $\Gamma^{\Sigma}_{p,d}(|E|)$ can be embedded into $\Gamma_{G(a,b)}$ as an induced subgraph. □

Using Claim 7 and the No-Homomorphism Lemma, we conclude that

$$1 \leq \rho(G(a,b)) = \frac{\alpha(\Gamma_{G(a,b)})}{\frac{|G(a,b)|}{3p}} \leq \frac{\alpha(\Gamma^{\Sigma}_{p,d}(|E|))}{\frac{|E|pd}{3p}} = \frac{\max\{|E_{x_1}|d, |E|\}}{|E_{x_1}|d} = \max\{1, \tfrac{3}{d}\}.$$

If $d \geq 3$, then clearly $\rho(G(a,b)) = 1$. If $1 \leq d \leq 2$, then $\frac{3}{d} = \frac{|K|}{|K_{x_1}|d}$ is attained through the intersecting set K. Thus, if $1 \leq d \leq 2$, then $\rho(G(a,b)) = \frac{3}{d}$. This completes the proof of the theorem. □

9. The doubly transitive case

Throughout this section, we assume the following.

Assumption 2. Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group satisfying Assumption 1 such that $\overline{G}$ is 2-transitive.

As $\overline{G}$ has degree p, the theory of transitive groups of prime degrees plays an important role in what follows. The next lemma is crucial to the proof of the main result of this section.

Lemma 9.1. *If $H \leq \mathrm{Sym}(p)$ is transitive and P is a Sylow p-subgroup of H, then P is cyclic and $\mathrm{N}_H(P) = P$ if and only if $H = P$.*

Proof. A Sylow p-subgroup of H has order p^k, for some $k \geq 1$. Since $p^2 \nmid p!$, clearly, a Sylow p-subgroup of any transitive subgroup of $\mathrm{Sym}(p)$ must be of order p and thus cyclic.

Let P be a Sylow p-subgroup of H. If $H = P$, then $\mathrm{N}_H(P) = H$. Conversely, if $\mathrm{N}_H(P) = P$, then P is the unique Sylow p-subgroup of H and $P \trianglelefteq H$. Hence, $P \leq H \leq \mathrm{N}_H(P) = P$, which completes the proof. $\square$

The main result of this section is the following.

Theorem 9.2. *If $G \leq \mathrm{Sym}(\Omega)$ satisfies Assumption 2, then $\rho(G) = 1$ unless $p = q+1$ is a Fermat prime and $\mathrm{PSL}_2(q) \leq \overline{G} \leq \mathrm{P\Gamma L}_2(q)$.*

Proof. As $\overline{G}$ is 2-transitive, it is one of the groups described in (a)-(e). Let $P \leq \overline{G}$ be a Sylow p-subgroup. By Lemma 9.1, P is cyclic. Since $\overline{G}$ is 2-transitive, the subgroup P cannot be self-normalizing. Therefore, $\mathrm{N}_{\overline{G}}(P) = P \rtimes Q$, for some cyclic subgroup $Q \leq \overline{G}$ such that $|Q| = d \neq 1$ and $d \mid (p-1)$.

Claim 8. *Q has an element of order at least 3, unless $\mathrm{PSL}_2(q) \leq \overline{G} \leq \mathrm{P\Gamma L}_2(q)$, where $p = q+1$ is a Fermat prime.*

Proof of Claim 8. If $\overline{G} = \mathrm{AGL}_1(p)$, then $\mathrm{N}_{\overline{G}}(P) = P \rtimes Q \cong \mathrm{AGL}_1(p)$, so $Q \cong C_{p-1}$ admits an element of odd order if $p-1$ is not a power of 2. Assume now that $p - 1 = 2^k$, for some $k \geq 1$. If $k \geq 2$, then it admits an element of order larger than 3. If $k = 1$, then $p = 3$ contradicts the fact that $p \geq 5$. This settles (a).

If $\overline{G} = \mathrm{Alt}(p)$, then $\mathrm{N}_{\overline{G}}(P) \cong C_p \rtimes C_{\frac{p-1}{2}}$. Similar to the previous paragraph, if $\frac{p-1}{2}$ is not a power of 2, then $Q \leq \mathrm{N}_{\overline{G}}(P)$ admits the desired element. If $p - 1 = 2^k$, for some $k \geq 3$, then a similar result holds. If $k = 2$, then Q does not have an element of order larger than 2. However, this case can be omitted from the analysis since no transitive group of degree 15 satisfies Assumption 1, and we already know that $\mathcal{I}_{15} = \{1\}$. This settles (b).

For (c), the normalizers of a Sylow 11-subgroup of $\mathrm{PSL}_2(11)$ and M_{11} are both isomorphic to $C_{11} \rtimes C_5$. Similarly, the normalizers of the groups in (d) are both isomorphic to $C_{23} \rtimes C_{11}$. Hence, the results follow trivially.

Finally, we consider the socle in (e), that is, $\mathrm{PSL}_n(q)$ for some prime number n and a prime power q, such that $p = \frac{q^n-1}{q-1}$. It is well known that $\mathrm{PSL}_n(q)$ admits a Singer cycle A of order p. Since a Singer subgroup (i.e., a subgroup generated by a Singer cycle) is isomorphic to a Sylow p-subgroup in this case, the normalizer of the Singer cycle $\langle A \rangle$ is a Frobenius group. Then by [12], we must have that n is an odd prime, or $n = 2$ and $4 \nmid (q+1)$. Assume that n is an odd prime. If $\Phi \in \mathrm{Aut}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is the Frobenius automorphism, then Φ induces a collineation B_Φ of $\mathrm{PG}_{n-1}(q)$. Then, $o(B_\Phi) = n$, and $\mathrm{N}_{\mathrm{PSL}_n(q)}(\langle A \rangle) = \langle A \rangle \rtimes \langle B_\Phi \rangle$. Thus, Q admits an element of order $n \geq 3$.

If $n = 2$, then it is well known that the normalizer of a cyclic group of order $p = q+1$, which is a Fermat prime, in $\mathrm{PSL}_2(q)$ is isomorphic to $\mathrm{D}_{2p} = C_p \rtimes C_2$, so $|Q| = 2$. $\square$

Now, let $M \leq G$ be such that $\overline{M} = P \rtimes Q = \langle \alpha \rangle \rtimes \langle \beta \rangle$. Clearly, M is transitive since $K \neq 1$, so $\rho(G) \leq \rho(M)$. If $a, b \in G$ such that $\overline{a} = \alpha \in P$ and $\overline{b} = \beta \in Q$, then

$$\rho(G) \leq \rho(M) = \rho(M(a,b)) = \max\{1, \tfrac{3}{|Q|}\}.$$

Therefore, we only need to show that $Q \leq \mathrm{N}_{\overline{G}}(P)$ contains an element of order at least 3 normalizing P to show that $\rho(G) = 1$. By Claim 8, we conclude that $\rho(G) = 1$ unless $\mathrm{PSL}_2(q) \leq \overline{G} \leq \mathrm{P\Gamma L}_2(q)$, where $p = q+1$ is a Fermat prime, in which case $1 \leq \rho(G) \leq \frac{3}{2}$. $\square$

10. Concluding remarks

In this paper, we showed that if p is an odd prime, then for any imprimitive group G of degree $3p$ which is not quasiprimitive (i.e., admitting a non-trivial and intransitive normal subgroup), $\rho(G) \in \{1, \frac{3}{2}, 3\}$, unless possibly when $p = 2^{2^k} + 1$ is a Fermat prime and the induced action of G on the unique G-invariant partition of Ω is an almost simple group containing a subgroup isomorphic to $\mathrm{PSL}_2(2^{2^k})$. For the aforementioned case, we can only give an upper bound of $\frac{3}{2}$ on the intersection density of G. We are inclined to believe that the intersection densities of these groups arising from Fermat primes $p = q+1$ and $\mathrm{PSL}_2(q)$ are equal to 1. Thus, we pose the following question.

Question 10.1. Let $p = q+1$ be a Fermat prime. Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group of degree $3p$ satisfying Assumption 1 such that $\mathrm{PSL}_2(q) \leq \overline{G} \leq \mathrm{P\Gamma L}_2(q)$. Is it true that $\rho(G) = 1$?

The results in this paper are further evidence of the veracity of Meagher's question in Question 1.3. Provided that Question 10.1 is affirmative, the only cases left to check are the quasiprimitive cases. It was proved in [3] that the only quasiprimitive groups of degree $3p$ whose intersection densities are possibly larger than 1 are almost simple groups with socle equal to $\mathrm{PSL}_n(q)$, where n is a prime, q is a prime power, and $p = \frac{q^n-1}{q-1}$ is an odd prime. In this case, $K = 1$, so we cannot apply the arguments used in this paper anymore.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

We are grateful to the reviewers for their insightful feedback. The majority of this work was completed when the authors were postdoctoral fellows at the University of Primorska, Faculty of Mathematics, Natural Sciences, and Information Technologies, Slovenia.

Data availability

No data was used for the research described in the article.

References

[1] B. Ahmadi, K. Meagher, The Erdős-Ko-Rado property for some permutation groups, Australas. J. Comb. 61 (1) (2015) 23–41.
[2] M.O. Albertson, K.L. Collins, Homomorphisms of 3-chromatic graphs, Discrete Math. 54 (2) (1985) 127–132.
[3] A. Behajaina, R. Maleki, A.S. Razafimahatratra, Intersection density of imprimitive groups of degree pq, J. Comb. Ser. A 208 (2024) 105922.
[4] P.J. Cameron, C.Y. Ku, Intersecting families of permutations, Eur. J. Comb. 24 (7) (2003) 881–890.
[5] M. Deza, P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Comb. Ser. A 22 (3) (1977) 352–360.
[6] J.D. Dixon, B. Mortimer, Permutation Groups, vol. 163, Springer Science & Business Media, 1996.
[7] D. Ellis, E. Friedgut, H. Pilpel, Intersecting families of permutations, J. Am. Math. Soc. 24 (3) (2011) 649–682.
[8] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Q. J. Math. 12 (1) (1961) 313–320.
[9] A. Ernst, K. Schmidt, Intersection theorems for finite general linear groups, Math. Proc. Camb. Philos. Soc. 175 (1) (2023) 129–160.
[10] C. Godsil, K. Meagher, A new proof of the Erdős–Ko–Rado theorem for intersecting families of permutations, Eur. J. Comb. 30 (2) (2009) 404–414.
[11] C. Godsil, K. Meagher, Erdős-Ko-Rado Theorems: Algebraic Approaches, Cambridge University Press, 2016.
[12] M.D. Hestenes, Singer groups, Can. J. Math. 22 (3) (1970) 492–513.
[13] A. Hujdurović, K. Kutnar, B. Kuzma, D. Marušič, Š. Miklavič, M. Orel, On intersection density of transitive groups of degree a product of two odd primes, Finite Fields Appl. 78 (2022) 101975.
[14] A. Hujdurović, K. Kutnar, D. Marušič, Š. Miklavič, Intersection density of transitive groups of certain degrees, Algebraic Combin. 5 (2) (2022) 289–297.
[15] C. Jordan, Recherches sur les substitutions, J. Math. Pures Appl. 17 (1872) 351–367.
[16] B. Larose, C. Malvenuto, Stable sets of maximal size in Kneser-type graphs, Eur. J. Comb. 25 (5) (2004) 657–673.
[17] C.H. Li, S.J. Song, V. Pantangi, Erdős-Ko-Rado problems for permutation groups, arXiv preprint, arXiv:2006.10339, 2020.
[18] D. Marušič, On vertex symmetric digraphs, Discrete Math. 36 (1) (1981) 69–81.
[19] K. Meagher, P. Spiga, An Erdős–Ko–Rado theorem for the derangement graph of $PGL(2,q)$ acting on the projective line, J. Comb. Ser. A 118 (2) (2011) 532–544.
[20] K. Meagher, A.S. Razafimahatratra, P. Spiga, On triangles in derangement graphs, J. Comb. Ser. A 180 (2021) 105390.
[21] A.S. Razafimahatratra, On complete multipartite derangement graphs, Ars Math. Contemp. 21 (1) (2021) 1–07.
[22] A.S. Razafimahatratra, On the intersection density of primitive groups of degree a product of two odd primes, J. Comb. Ser. A 194 (2023) 105707.
[23] P. Spiga, The Erdős-Ko-Rado theorem for the derangement graph of the projective general linear group acting on the projective space, J. Comb. Ser. A 166 (2019) 59–90.
[24] The Sage Developers, SageMath, the Sage Mathematics Software System (Version 10.4), 2024, https://www.sagemath.org.