Linear Algebra and its Applications 610 (2021) 86–119

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Weakly Hadamard diagonalizable graphs

Mohammad Adm [a], Khawla Almuhtaseb [a], Shaun Fallat [b,*], Karen Meagher [b], Shahla Nasserasr [c], Mahsa N. Shirazi [b], A.S. Razafimahatratra [b]

[a] *Department of Applied Mathematics and Physics, Palestine Polytechnic University, Hebron, Palestine*
[b] *Department of Mathematics and Statistics, University of Regina, Regina, SK, S4S 0A2, Canada*
[c] *School of Mathematical Sciences, Rochester Institute of Technology, Rochester, NY, 14623, USA*

A R T I C L E   I N F O

Article history:
Received 14 April 2020
Accepted 25 September 2020
Available online 1 October 2020
Submitted by R. Brualdi

MSC:
05C50
15A18

Keywords:
Hadamard matrices
Laplacians
Eigenspaces
Strongly-regular graphs

A B S T R A C T

A matrix is called weakly Hadamard if its entries are from $\{0, -1, 1\}$ and its non-consecutive columns (with some ordering) are orthogonal. Unlike Hadamard matrices, there is a weakly Hadamard matrix of order n for every $n \geq 1$. In this work, graphs for which their Laplacian matrices can be diagonalized by a weakly Hadamard matrix are studied. A number of necessary and sufficient conditions are verified along with identification of numerous families of graphs whose Laplacian matrices can be diagonalized by a weakly Hadamard matrix.

1. Introduction

The *Laplacian matrix* of a graph G on n vertices is an $n \times n$ matrix $L(G)$ such that its (i, j) entry for $i \neq j$ equals -1 if the vertices i and j are adjacent, the (i, i) entry

* Corresponding author.
E-mail addresses: khawla@ppu.edu (K. Almuhtaseb), shaun.fallat@uregina.ca (S. Fallat).

https://doi.org/10.1016/j.laa.2020.09.038
0024-3795/

equals the degree of the vertex i in G, and all other entries are 0. An $n \times n$ matrix is a *Hadamard matrix of order* n if its entries are equal to either 1 or -1, and

$$H^t H = nI_n.$$

One of the interesting questions in the spectral graph theory is about the structure of the eigenvectors of matrices associated with graphs. Barik, Fallat and Kirkland in [3] studied graphs for which their Laplacian matrix can be diagonalized by a Hadamard matrix H. More precisely, they considered graphs on n vertices such that their Laplacian matrix has n orthogonal eigenvectors with entries from the set $\{-1, 1\}$. A graph with this property is called a *Hadamard diagonalizable graph*. It turns out that there is a natural and fruitful connection between Hadamard diagonalizable graphs and graphs possessing perfect quantum state transfer, see, for instance, [5,11]. A connection was also made between balancedly splittable Hadamard matrices and Hadamard diagonalizable strongly regular graphs in [12]. Here we extend and expand upon the results in [3] by introducing zero to the entries of the eigenvectors as well as relaxing orthogonality condition among vectors within eigenspaces.

If the entries are restricted to real numbers, it is well-known that if H is an $n \times n$ Hadamard matrix, then n is 1, 2, or a multiple of 4. However, it is not known if there exists a Hadamard matrix of order n for $n = 4k, k \geq 1$. The Hadamard matrix conjecture, sometimes called Paley's conjecture, states that for every $n = 4k, k \geq 1$ there exists a Hadamard matrix of order n. A well-known method of constructing Hadamard matrices is *Sylvester's construction*. This construction starts with the matrix

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and then for any positive integer k defines the Hadamard matrix $H_{2^k} = H_2 \otimes H_{2^{k-1}}$, where the operation $\otimes$ denotes the tensor (or Kronecker) product of matrices. The matrices produced by this construction are called *Sylvester Hadamard*. This construction implies that there is a Hadamard matrix of order 2^k for all $k \geq 1$. Additionally, Paley showed the existence of one of the largest classes of Hadamard matrices, those of order $1+p$ and $2(1+p)$ for prime powers p, with $p \equiv 3 \ (mod\ 4)$ and $p \equiv 1 \ (mod\ 4)$. There are other sporadic examples of Hadamard matrices. For instance, it is not difficult to see that if H_1 and H_2 are both Hadamard matrices then their tensor product $H_1 \otimes H_2$ is also a Hadamard matrix.

The absence of definitive knowledge about the existence of Hadamard matrices makes characterizing graphs that are Hadamard diagonalizable challenging. In fact, a complete graph on n vertices is Hadamard diagonalizable if and only if a Hadamard matrix of order n exists [3]. This means that determining which complete graphs are Hadamard diagonalizable requires first demonstrating that a Hadamard matrix of a given order n exists, and hence resolving a famous open problem (the Hadamard conjecture).

In this work, we generalize the notion of Hadamard matrices and introduce a family of matrices with two properties: 1) the entries of the matrix are from the set $\{-1, 0, 1\}$;

2) there is an ordering of the columns of the matrix so that the non-consecutive columns are orthogonal. The consecutive columns can be either orthogonal or not orthogonal. The second condition implies that the product of any such matrix with its transpose is a tridiagonal matrix. We call a matrix with these two properties a *weakly Hadamard diagonalizable* matrix and denote it by WHD. We investigate graphs for which their Laplacian matrix can be diagonalized by a weakly Hadamard matrix. We note here that investigating structured eigenbases associated with specific matrices has occurred previously and is of interest to the community, see for example, [1].

Definition 1.1. A graph is *weakly Hadamard diagonalizable* if its Laplacian matrix L can be diagonalized with a weakly Hadamard matrix. In other words, if L can be written as $L = PDP^{-1}$, where D is a diagonal matrix and P has the properties that all the entries of P are from $\{-1, 0, 1\}$ and that P^tP is a tridiagonal matrix.

Clearly, any Hadamard diagonalizable graph is also weakly Hadamard diagonalizable. However, the converse need not hold in general.

Example 1.2. Let X be the complete graph K_4 minus one edge. Then $L(X)$ is weakly Hadamard diagonalizable by the following matrix P.

$$L(X) = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix}.$$

However, X is not Hadamard diagonalizable as it is not a regular graph.

Consider a set of vectors $B = \{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$. We say the vectors in B are *quasi-orthogonal* if there is an ordering of the vectors of B such that non-consecutive vectors are orthogonal. So if P is a matrix whose columns are the vectors of B, in the given ordering, then P^tP is a tridiagonal matrix. Since eigenvectors corresponding to distinct eigenvalues are orthogonal, in our approach to find graphs that are WHD, it is sufficient to find a quasi-orthogonal basis for each eigenspace associated with distinct eigenvalues of the Laplacian matrix.

Proposition 1.3. *A graph is WHD if there exists a quasi-orthogonal basis for each eigenspace in which the entries of every vector are from* $\{-1, 0, 1\}$.

Proposition 1.4. *If X is a regular graph then X is WHD if and only if the adjacency matrix of X has a basis of quasi-orthogonal eigenvectors with all entries from* $\{-1, 0, 1\}$.

Throughout this paper, $\mathbf{1}$ denotes the all ones vector, and e_i denotes the standard basis vector; i.e. every entry is equal to zero except the ith entry which is equal to one. The $m \times m$ identity matrix is denoted by I_m, and J_m is the $m \times m$ all ones matrix. The dimension of these vectors will be clear from context. A vector $\chi = [\chi_i]$ in $\mathbb{R}^n$ will be called a characteristic vector for a set $S \subset \{1, 2, \dots, n\} = [n]$, if

$$\chi_i = \begin{cases} 1, & \text{if } i \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose A is an $n \times n$ matrix, we use $\sigma(A)$ to denote the spectrum of A. Let λ be an eigenvalue for A. We refer to an eigenvector associated with λ as a λ-eigenvector, and an eigenbasis for the eigenspace associated with λ as a λ-eigenspace, and may denote such a basis by E_λ.

First we show that the complete graph K_n is WHD for every value of $n \geq 1$. Note that this is not the case in the usual Hadamard diagonalizable graphs.

Lemma 1.5. *For every integer $n \geq 1$ the graph K_n is WHD.*

Proof. The complete graph K_n is $(n-1)$-regular, thus $\mathbf{1}$ is an eigenvector of $L(K_n) = nI_n - J_n$ corresponding to the eigenvalue 0. Further, the vectors $v_i = e_i - e_{i+1}$ for $i \in \{1, \dots, n-1\}$ form a basis for the eigenspace corresponding to the eigenvalue n. The vectors v_i and v_j are orthogonal for every i and j with $i - j > 1$, this completes the proof. $\square$

In this paper we first give some basic results for WHD graphs. We show some of the differences and similarities between Hadamard diagonalizable graphs and WHD graphs. In Section 3 we provide conditions on graphs that are sufficient in order to produce WHD graphs using products of graphs. In Section 4, we show that join of any number of WHD graphs is WHD if their sizes satisfy in a newly defined partition called recursively balanced partition. We also provide more families of joins of graphs such as a complete graph minus a matching that are WHD. In Section 5, we show that several families of strongly-regular graphs are WHD. Finally, we list some interesting questions that remain open and provide a complete list of all graphs on at most nine vertices that are WHD in an Appendix.

2. Basic properties of WHD graphs

In this section we extend some of the existing results on Hadamard diagonalizable graphs to weakly Hadamard diagonalizable graphs. For two graphs X and Y, their disjoint union is denoted by $X \sqcup Y$, and the complement of X is denoted by X^c.

Lemma 2.1. *[3] If X is a Hadamard diagonalizable graph then*

(1) *X is regular;*
(2) *all of the Laplacian eigenvalues of X are even integers;*
(3) *$X \sqcup X$ is Hadamard diagonalizable;*

(4) *X^c is Hadamard diagonalizable.*

Example 1.2 shows that Part 1 of Lemma 2.1 need not hold in general for WHD graphs. We show that a relaxed version of Part 2 of Lemma 2.1 holds for WHD graphs.

Lemma 2.2. *If X is WHD, then all the Laplacian eigenvalues of X are integers.*

Proof. If X is WHD, then there is a basis of eigenvectors of $L(X)$ with all entries from the set $\{0, -1, 1\}$. For any eigenvalue λ, choose any such eigenvector y. Then from the eigen-equation $L(X)y = \lambda y$, it follows that $L(X)y$ must be integral, and hence λ must be an integer, since y has entries $\{0, -1, 1\}$ and is nonzero. □

Using Proposition 2.7 below, the cycle C_6 is WHD and its Laplacian eigenvalues are $\{0, 1, 1, 3, 3, 4\}$ hence an example where some of the eigenvalues are odd integers.

Similarly, a more general version of Part 3 of Lemma 2.1 holds for WHD graphs.

Lemma 2.3. *If X and Y are WHD graphs, then $X \sqcup Y$ is also a WHD graph.*

Proof. Since X and Y are WHD, there is a basis for X (resp. Y) of v_i (resp. w_i) that satisfy Proposition 1.3. Then $[1, 0]^t \otimes v_i$ and $[0, 1]^t \otimes w_i$ also satisfy the conditions of Proposition 1.3 for $X \sqcup Y$. □

Lemma 9 of [3] implies that $K_4 \sqcup K_8$ is not Hadamard diagonalizable. Therefore, Lemma 2.3 is not true for Hadamard diagonalizable graphs.

Finally, Part 4 of Lemma 2.1 can be extended to WHD, but an extra condition is needed on the graph.

Lemma 2.4. *If X is a connected WHD graph, then X^c is also a WHD graph.*

Proof. Assume that X is a connected WHD graph on n vertices. Let v_i be an eigenvector for $L(X)$ with the eigenvalue λ_i. Assume that $P^{-1}L(X)P$ is a diagonal matrix and P^tP is tridiagonal for the matrix P with columns $v_1 = \mathbf{1}, v_2, \ldots, v_n$. Since X is connected, v_i is orthogonal to $\mathbf{1}$ for $i = 2, \ldots, n$.

Then $L(X^c) = nI - L(X) - J$, so

$$L(X^c)v_1 = (nI - L(X) - J)\mathbf{1} = (n - 0 - n)\mathbf{1} = 0\mathbf{1} = 0,$$
$$L(X^c)v_i = (nI - L(X) - J)v_i = (n - \lambda_i - 0)v_i = (n - \lambda_i)v_i.$$

This means that the columns of P are also eigenvectors for $L(X^c)$, so X^c is WHD with P. □

If X is disconnected and WHD, then X^c is not necessarily a WHD graph. A simple such example illustrating this claim is $X = K_1 \sqcup K_2$. Observe that X is WHD by

Lemma 2.3, but $X^c = P_3$, the path on 3 vertices, and $L(P_3)$ has eigenvalues $0, 1 \pm \sqrt{2}$, and so Lemma 2.2 implies X is not WHD.

The previous proof required that all eigenvectors, other than the all ones vector, be orthogonal to $\mathbf{1}$. This can be achieved in a regular disconnected graph if all the components have the same size.

Lemma 2.5. *Assume that X is a disconnected WHD graph on n vertices. If X is regular and its components are of equal size, then X^c is also a WHD graph.*

Proof. Assume that the WHD graph X has k components $G_1, G_2, \dots, G_k$ of equal size. Thus, there is a diagonalizable basis of eigenvectors of $L(X)$, say $v_1, \dots, v_n$, with entries from $\{0, -1, 1\}$. Let w_i be the characteristic vector for the component G_i, $i = 1, \dots, k$. Then the set of vectors $w_{i-1} - w_i$, where $i = 2, 3, \dots, k$, along with $\mathbf{1}$ form a quasi-orthogonal basis of the eigenspace corresponding to the eigenvalue zero.

As in the previous proof, each of $v_{k+1}, \dots, v_n$ are eigenvectors for $L(X^c)$. Thus there is a quasi-orthogonal basis for each eigenspace. $\square$

Lemmas 1.5 and 2.5 imply that

$$K_{n,n} = (K_n \sqcup K_n)^c$$

is WHD. However, the following lemma shows that Lemmas 2.5 and 2.4 cannot be extended to all bipartite graphs.

Lemma 2.6. *The graph $K_{n,m}$ is WHD if and only if $m = n$.*

Proof. Since $K_{n,n} = (K_n \sqcup K_n)^c$, Lemmas 1.5 and 2.5 show that $K_{n,n}$ is WHD.

If $n \neq m$, then $n + m$ is an eigenvalue for $K_{n,m}$ with multiplicity 1. If the vertices of $K_{n,m}$ are ordered so that the n vertices with degree m occur first, then the $(n+m)$-eigenspace is spanned by the vector with the first n entries equal to m, and the remaining m entries equal to $-n$. Since $m \neq n$ there is no vector in this eigenspace with entries from the set $\{0, -1, 1\}$. Thus, by Proposition 1.3, $K_{n,m}$ is not WHD when $n \neq m$. $\square$

We end this section with a note about which cycles are weakly Hadamard diagonalizable.

Proposition 2.7. *The cycle C_n is WHD if and only if $n = 3, 4$ or 6.*

Proof. The eigenvalues of the adjacency matrix of the cycle C_n are of the form

$$\omega_k = 2\cos\left(\frac{2k\pi}{n}\right), \text{ for any } k = 0, 1, 2, \dots, n-1.$$

For C_n to have integral spectrum, we must have $\cos\left(\frac{2k\pi}{n}\right) \in \{0, \pm 1, \pm \frac{1}{2}\}$. The only roots of unity that verify this property are the 3-rd, 4-th and 6-th roots of unity. Now we prove that for $n = 3, 4, 6$, the cycle C_n is WHD.

The cycle C_3 is WHD because it is a complete graph. Since $C_4 = K_{2,2}$, it is WHD by Lemma 2.6.

For C_6, the Laplacian eigenvalues are $\{0^{(1)},\ 1^{(2)}, 3^{(2)}, 4^{(1)}\}$ and the columns of the following form a quasi-orthogonal eigenbasis

$$P = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 \end{pmatrix}.$$

Therefore, C_6 is WHD. $\square$

3. Graph products

In [3], it is shown that for many graph products, if the constituent graphs are Hadamard diagonalizable, then the product graph is also Hadamard diagonalizable. In this section we extend these results to WHD graphs for the following graph products. Let X and Y be graphs.

(1) The *Cartesian product* of X and Y, denoted $X \,\square\, Y$ is the graph with vertex set $V(X) \times V(Y)$ and

$$(u_1, v_1) \sim_{X \,\square\, Y} (u_2, v_2) \iff \begin{cases} u_1 = u_2 \text{ or } v_1 \sim_Y v_2, \\ v_1 = v_2 \text{ or } u_1 \sim_X u_2. \end{cases}$$

(2) The *direct product* (tensor product) of X and Y, denoted $X \times Y$, is the graph with vertex set $V(X) \times V(Y)$ and

$$(u_1, v_1) \sim_{X \times Y} (u_2, v_2) \iff u_1 \sim_X u_2 \text{ and } v_1 \sim_Y v_2.$$

(3) The *strong product* $X \boxtimes Y$ of the graph X and Y is the graph with vertex-set $V(X) \times V(Y)$ such that

$$(u_1, v_1) \sim_{X \boxtimes Y} (u_2, v_2) \iff \begin{cases} u_1 = u_2 \text{ or } v_2 \sim_Y v_2, \\ v_1 = v_2 \text{ or } u_1 \sim_X u_2, \\ u_1 \sim_X u_2 \text{ and } v_1 \sim_Y v_2. \end{cases}$$

For each of these products, if the graphs X and Y are regular, then the product graph is also regular and it is straightforward to calculate its degree. Further, the eigenvectors for the Laplacian matrix of each graph product can be calculated from the eigenvectors of the constituents.

Lemma 3.1. *Consider graphs X and Y on n and m vertices, respectively. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $L(X)$, and $\mu_1, \mu_2, \dots, \mu_m$ be the eigenvalues of $L(Y)$. Further assume that u_i is an eigenvector of $L(X)$ for eigenvalue λ_i, and v_j an eigenvector of $L(Y)$ corresponding to the eigenvalue μ_j,*

(1) *The eigenvalues of $L(X \,\square\, Y)$ are $\lambda_i + \mu_j$, for $1 \le i \le n$ and $1 \le j \le m$. Moreover, $u_i \otimes v_j$ is an eigenvector of $L(X \,\square\, Y)$ for the eigenvalue $\lambda_i + \mu_j$.*
(2) *The eigenvalues of $L(X \times Y)$ are $\lambda_i \mu_j$ for any $1 \le i \le n$ and $1 \le j \le m$. Moreover, $u_i \otimes v_j$ is an eigenvector of $L(X \times Y)$ corresponding to the eigenvalue $\lambda_i \mu_j$.*
(3) *The eigenvalues of $L(X \boxtimes Y)$ are of the form $(\lambda_i + 1)(\mu_j + 1) - 1$, for $1 \le i \le n$ and $1 \le j \le m$. Again, the eigenvectors are of the form $u_i \otimes v_j$.*

Since we have the eigenvectors for each of these graph products, we can see that if the constituent graphs are Hadamard diagonalizable, then the product graph is also Hadamard diagonalizable.

Theorem 3.2. *[3] If X and Y are both Hadamard diagonalizable graphs of order n and m respectively then $X \star Y$ is Hadamard diagonalizable for $\star \in \{\,\square\,, \boxtimes, \times\}$.*

The proofs for these results are straightforward. If H_1 and H_2 are Hadamard matrices that diagonalize X and Y respectively, then $H_1 \otimes H_2$ is also a Hadamard matrix and it diagonalizes the graph product. This argument cannot be generalized to WHD graphs. Consider WHD graphs X and Y with matrices P_1 and P_2 that diagonalize X and Y (respectively) such that $P_1^t P_1$ and $P_2^t P_2$ are both tridiagonal. The matrix $P_1 \otimes P_2$ diagonalizes the graph $X \,\square\, Y$ (as well as $X \times Y$ and $X \boxtimes Y$), but $(P_1 \otimes P_2)^t (P_1 \otimes P_2)$ is not necessarily tridiagonal. So an extra condition is required to guarantee that the graph products produce WHD graphs.

Proposition 3.3. *Suppose graphs X and Y are WHD with the matrix of eigenvectors P_X and P_Y, respectively, such that $P_X^t P_X$ is a diagonal matrix. Then $X \star Y$ is WHD for any $\star \in \{\,\square\,, \boxtimes, \times\}$.*

Proof. Since P_X and P_Y have entries $\{0, -1, 1\}$, so does $P = P_X \otimes P_Y$. If $P_X^t P_X = D$ is a diagonal matrix, then $P^t P = P_X^t P_X \otimes P_Y^t P_Y = D \otimes P_Y^t P_Y$ is tridiagonal. Hence $X \star Y$ is WHD with the weakly Hadamard matrix P. $\square$

Corollary 3.4. *If X is Hadamard diagonalizable and Y is a WHD graph, then $X \star Y$ is WHD for any $\star \in \{\,\square\,, \boxtimes, \times\}$.*

Note that Proposition 3.3 does not characterize the products of graphs that are WHD. For example we know that $K_{nm} = K_n \boxtimes K_m$ is WHD, but this graph is not included in Proposition 3.3. In particular, the matrix $P = P_{K_n} \otimes P_{K_m}$, has entries from $\{0, -1, 1\}$ and it diagonalizes K_{nm}, but it does not give a tridiagonal matrix when multiplied by its transpose on the right. So this natural construction of eigenvectors with entries from $\{0, -1, 1\}$ for the product graph may not give a quasi-orthogonal basis of eigenvectors.

4. Joins of graphs

Let X_1 and X_2 be graphs on n_1 and n_2 vertices, respectively. The *join* of X_1 and X_2, denoted by $X_1 \vee X_2$, is the graph formed by taking the union of X_1 and X_2 and adding every edge between the vertices in X_1 and the vertices in X_2. Assume $\sigma(L(X_1)) = \{0 = \lambda_1, \dots, \lambda_{n_1}\}$ with eigenvectors $\{v_j^1\}_{j=1}^{n_1}$ and $\sigma(L(X_2)) = \{0 = \mu_1, \dots, \mu_{n_2}\}$ with eigenvectors $\{v_j^2\}_{j=1}^{n_2}$. Then $\sigma(L(X_1 \vee X_2)) = \{0, n_1 + n_2, \lambda_2 + n_2, \dots, \lambda_{n_1} + n_2, \mu_2 + n_1, \dots, \mu_{n_2} + n_1\}$. The eigenvectors of $L(X_1 \vee X_2)$ are $\mathbf{1}$ for the eigenvalue 0; $e_i \otimes v_j^i$ with $i = 1, 2$ and $j \neq 1$ for the eigenvalues other than 0 and $n_1 + n_2$. The eigenvector corresponding to the eigenvalue $n_1 + n_2$ is a vector where its first n_1 entries are equal to n_2 and the last n_2 entries are equal to $-n_1$. More generally, an eigenvector of $L(X_1 \vee X_2 \vee \cdots \vee X_k)$ with $k \geq 3$ corresponding to the eigenvalue $\sum_{i=1}^{k} n_i$ can be of the form $(e_i \otimes n_j \mathbf{1}) - (e_j \otimes n_i \mathbf{1})$ for $i, j \in \{1, 2, \dots, k\}$.

The following result for Hadamard diagonalizable graphs can be improved in the case of WHD graphs.

Lemma 4.1. *[3, Lemma 7] If X is a Hadamard diagonalizable graph, then $X \vee X$ is also a Hadamard diagonalizable graph.*

We define an integer partition of an integer n to be *recursively balanced partition* if it satisfies the following. The partition with only one part, i.e. $P = [n]$, is defined to be a recursively balanced partition. A partition $P = [n_1, n_2, \dots, n_k], k \geq 2$ is called a recursively balanced partition if there is a partition $Q = [Q_1, Q_2, \dots, Q_\ell]$ of the parts of P with $Q_i = [n_{i_1}, \dots, n_{i_{k_i}}]$ such that

(1) for any $i, j \in \{1, \dots, \ell\}$

$$n_{i_1} + n_{i_2} + \cdots + n_{i_{k_i}} = n_{j_1} + n_{j_2} + \cdots + n_{j_{k_j}},$$

and

(2) each sub-partition Q_i is also a recursively balanced partition.

For example, the recursively balanced partitions of 8 are:

$$[8], [4, 4], [4, 2, 2], [4, 2, 1, 1], [4, 1, 1, 1, 1], [2, 2, 2, 2], [2, 2, 2, 1, 1],$$
$$[2, 2, 1, 1, 1, 1], [2, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1].$$

Proposition 4.2. *If $P = [n_1, n_2, \dots, n_k]$ is a recursively balanced partition, then there are $k - 1$ equations of the form*

$$n_{i_1} + n_{i_2} + \cdots + n_{i_m} = n_{j_1} + n_{j_2} + \cdots + n_{j_n}.$$

For example, the partition $[4, 2, 1, 1]$ has the equations

$$4 = 2 + 1 + 1, \quad 2 = 1 + 1, \quad 1 = 1.$$

Lemma 4.3. *Let X_i for $i = 1, \ldots, k$ be connected WHD graphs on n_i vertices. If $[n_1, n_2, \ldots, n_k]$ is a recursively balanced partition, then $\bigvee_{i=1}^{k} X_i$ is a WHD graph.*

Proof. Assume that X_i is a WHD graph on n_i vertices, for $i \in \{1, \ldots, k\}$, and $[n_1, n_2, \ldots, n_k]$ is a recursively balanced partition. For each X_i, with $i \in \{1, \ldots, k\}$, let v_j^i for $j \in \{1, \ldots, n_i\}$ be a set of eigenvectors of $L(X_i)$ with entries from $\{0, -1, 1\}$ that are quasi-orthogonal with the given ordering. For each i, assume that $v_1^i = \mathbf{1}$ and that the eigenvalue for v_j^i is λ_j^i.

We construct a set of eigenvectors for $L(\bigvee_{i=1}^{k} X_i)$ with entries from $\{0, -1, 1\}$, and show that they satisfy the quasi-orthogonal property. The eigenvalue 0 has the eigenvector $\mathbf{1}$, and the eigenvalues $\lambda_j^i + \sum_{i \neq j} n_i$ have eigenvectors $e_i \otimes v_j^i$, for $j \neq 1$. These vectors are clearly linearly independent, have entries from $\{0, -1, 1\}$ and have the quasi-orthogonal property.

Now $[n_1, n_2, \ldots, n_k]$ is a recursive balanced partition; this means that there are exactly $k - 1$ equations of the form

$$n_{i_1} + n_{i_2} + \cdots + n_{i_m} = n_{j_1} + n_{j_2} + \cdots + n_{j_n}.$$

For each equation, define a vector

$$v = (1) \sum_{\ell=1}^{m} (e_{i_\ell} \otimes \mathbf{1}) + (-1) \sum_{\ell=1}^{n} (e_{j_\ell} \otimes \mathbf{1})$$

This vector is a linear combination of vectors of the form $e_i \otimes n_j \mathbf{1} - e_j \otimes n_i \mathbf{1}$, so it is in the $\sum_i n_i$-eigenspace. Further these vectors are orthogonal since the partitions are either disjoint, or refinements. □

We can apply this to complete multipartite graphs, since they are joins of empty graphs.

Corollary 4.4. *The complete bipartite $K_{n,n}$ is WHD. The complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ is WHD if $[n_1, n_2, \ldots, n_k]$ is a recursively balanced partition.*

Note that the 12th graph in the appendix on 8 vertices (this is the graph $K_{3,2,2,1}$) shows that the condition in the previous corollary does not characterize complete multipartite graphs that are WHD.

Similar to the case for the complete graphs, less can be said about when a complete bipartite graph is a Hadamard diagonalizable graph.

Lemma 4.5. *If there is a Hadamard matrix of order n, then the complete bipartite graph $K_{n,n}$ is Hadamard diagonalizable.*

Proof. Let H be a Hadamard matrix of order n. Without loss of generality we can consider the first row and column of H to be all ones vectors. Then

$$\begin{pmatrix} H & H \\ H & -H \end{pmatrix}\begin{pmatrix} nI & -J \\ -J & nI \end{pmatrix}\begin{pmatrix} H^t & H^t \\ H^t & -H^t \end{pmatrix} = diag(0, 2n^2, \dots, 2n^2, 4n^2, 2n^2, \dots, 2n^2). \quad \square$$

For a given symmetric matrix A, we denote the spectrum of A by $\sigma(A) = \{\lambda_1^{(n_1)}, \lambda_2^{(n_2)}, \dots, \lambda_\ell^{(n_\ell)}\}$, where $\lambda_i \neq \lambda_j$ when $i \neq j$, and where n_j denotes the multiplicity of the eigenvalue λ_j.

Lemma 4.6. *Let $X = \overline{K_k} \vee K_n$. If $n - k \in \{0, 1, 2\}$, then X is a WHD graph.*

Proof. We have $\sigma(L(X)) = \{0^{(1)}, n^{(k-1)}, (n+k)^{(n)}\}$. The all ones vector is an eigenvector for 0.

Order the vertices of X so that the vertices from $\overline{K_k}$ are the first k vertices. Then the vectors $e_i - e_{i+1}$, with $i = 1, \dots, k-1$, are suitable eigenvectors for the n-eigenspace. Similarly, the vectors $e_i - e_{i+1}$ with $i = k, \dots, n+k-1$ are $n-k-2$ suitable eigenvectors for the $(n+k)$-eigenspace. The vector $v = (1, 1, \dots, 1, -1, -1, \dots, -1, n-k-1)$ where the first k entries are equal to 1, the next $n-k-1$ entries are -1, and the last entry is equal to $n-k-1$ is an $(n+k)$-eigenvector. Moreover, if $n-k \in \{0, 1, 2\}$, then the eigenvector v has entries from $\{1, 0, -1\}$. $\square$

Lemma 4.7. *Let $X = H \vee K_n$ where H is a WHD connected graph on k vertices. If $n - k \in \{0, 1, 2\}$, then X is a WHD graph.*

Proof. Let $\sigma(L(H)) = \{0, \lambda_1^{(n_1)}, \lambda_2^{(n_2)}, \dots, \lambda_\ell^{(n_\ell)}\}$, then

$$\sigma(L(X)) = \{0^{(1)}, (n+\lambda_1)^{(n_1)}, (n+\lambda_2)^{(n_2)}, \dots, (n+\lambda_\ell)^{(n_\ell)}, (n+k)^{(n)}\}.$$

The all ones vector is an eigenvector for 0.

Order the vertices of X so that the vertices from H are the first k vertices. Then denote the λ_i-eigenvectors that form a weakly Hadamard matrix which diagonalizes H by v_{i_j}. These vectors, concatenated with n ones form suitable eigenvectors for the $(n+\lambda_i)$-eigenspace.

Similarly, the vectors $e_i - e_{i+1}$ with $i = k, \dots, n+k-1$ are $n-k-2$ suitable eigenvectors for the $(n+k)$-eigenspace. The vector $v = (1, 1, \dots, 1, -1, -1, \dots, -1, n-k-1)$ where the first k entries are equal to 1, the next $n-k-1$ entries are -1, and the last entry is equal to $n-k-1$ is an $(n+k)$-eigenvector. If $n-k \in \{0, 1, 2\}$, then the vector v eigenvector has entries from $\{1, 0, -1\}$. $\square$

We note that $H \vee K_n = K_{2n-i} \backslash H^c$ if H has at least two vertices and $i \in \{0, 1, 2\}$. So the previous result can be seen as either a statement about the join of two graphs, or a statement about removing a subgraph from a complete graph.

In the next two results, we provide other examples of the join of graphs that is WHD; in these examples one of the graphs is disconnected. Note that the complete graph K_n minus s independent edges (matching) with $s \leq \dfrac{n}{2}$ can be written as $(K_1 \cup K_1) \vee (K_1 \cup K_1) \vee \cdots \vee (K_1 \cup K_1) \vee K_{n-2s}$. The following two results show that K_n minus a matching is WHD for $n \geq 4$. Observe, that if $n = 3$, then K_3 minus an edge is P_3, which is not WHD.

Lemma 4.8. *For $n \geq 4$, the graph G obtained from K_n minus an edge is WHD.*

Proof. Without loss of generality, consider the edge $\epsilon = \{1, 2\}$ and let $G = K_n - \epsilon$. Then the Laplacian spectrum of G is given by: eigenvalue 0 of multiplicity 1, with eigenvector $\mathbf{1}$; eigenvalue $n-2$ of multiplicity 1, with eigenvector $e_1 - e_2$; and finally eigenvalue n with multiplicity $n-2$, with eigenbasis: $b_i = e_1 + e_2 - 2e_i$, for $i = 3, 4, \ldots, n$.

We construct an eigenbasis with entries from $\{0, -1, 1\}$ for the eigenvalue $n-2$ as follows:

For n even ($n = 2(k+1)$), consider:

$$x_1 = \left[\begin{array}{r} 1 \\ 1 \\ \hline -1 \\ -1 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \end{array}\right], x_2 = \left[\begin{array}{r} 0 \\ 0 \\ \hline 1 \\ 1 \\ \hline -1 \\ -1 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \end{array}\right], \ldots, x_k = \left[\begin{array}{r} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 1 \\ 1 \\ \hline -1 \\ -1 \end{array}\right]$$

and

$$y_1 = \left[\begin{array}{r} 0 \\ 0 \\ \hline 1 \\ -1 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \end{array}\right], y_2 = \left[\begin{array}{r} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \\ -1 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \end{array}\right], \ldots, y_k = \left[\begin{array}{r} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \\ -1 \end{array}\right].$$

For n odd ($n = 2(l+1)+1$), consider:

$$x_1 = \left[\begin{array}{r} 1 \\ 1 \\ \hline -1 \\ -1 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right], x_2 = \left[\begin{array}{r} 0 \\ 0 \\ \hline 1 \\ 1 \\ \hline -1 \\ -1 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right], \ldots, x_l = \left[\begin{array}{r} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 1 \\ 1 \\ \hline -1 \\ -1 \\ \hline 0 \end{array}\right]$$

and

$$y_1 = \left[\begin{array}{r} 0 \\ 0 \\ \hline 1 \\ -1 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right], y_2 = \left[\begin{array}{r} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \\ -1 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right], \ldots, y_l = \left[\begin{array}{r} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \\ -1 \\ \hline 0 \end{array}\right], \text{ and } z = \left[\begin{array}{r} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline \vdots \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 1 \\ \hline -1 \end{array}\right].$$

In the even case observe that $\{y_1, y_2, \ldots, y_k\}$ forms a mutually orthogonal set and x_i is orthogonal to $\{x_{i+2}, x_{i+3}, \ldots, x_k\}$ for $i = 1, \ldots, k-2$. Finally observe that x_i is orthogonal to y_j for any i, j. Hence if we form the matrix $M = [x_1, x_2, \ldots x_k | y_1, y_2, \ldots y_k]$, then it follows that $M^t M$ is a tridiagonal matrix.

Similarly, in the odd case we have that $\{y_1, y_2, \ldots, y_l\}$ forms a mutually orthogonal set and x_1 is orthogonal to $\{x_3, x_4, \ldots x_l\}$, x_2 is orthogonal to $\{x_4, x_5, \ldots x_l\}$, and so-on x_{l-2} is orthogonal to x_l, and that x_i is orthogonal to y_j for any i, j. Finally, note that z is orthogonal to $\{x_1, x_2, \ldots, x_{l-1}, y_1, y_2, \ldots, y_{l-1}\}$, but z is not orthogonal to either x_l nor y_l. In this case we form the matrix $M = [x_1, x_2, \ldots x_{l-1} | x_l, z, y_l | y_1, y_2, \ldots y_{l-1}]$ and it follows that $M^t M$ is a tridiagonal matrix. $\square$

If we wish to delete two independent edges from K_n, then we require $n \geq 6$. Although it is true that the result holds for both $n = 3, 4$, it does not hold for $n = 5$. In the latter case an eigenbasis for the eigenvalue n is given by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{bmatrix},$$

and it can be easily checked that it is not possible to construct an eigenbasis with entries from $\{0, -1, 1\}$ for n in this case.

More generally, when we delete an edge from a graph we perturb the existing Laplacian matrix by adding the matrix

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

direct summed with the appropriate zero matrix. It is easy to see that the eigenvalues for this 2×2 matrix are 0, with eigenvector $\mathbf{1}$, and -2 with eigenvector $a_1 = e_1 - e_2$. Using this basic fact, we can deduce the eigenvectors for the Laplacian of the complete graph with a matching removed.

For $n \geq 6$, consider the Laplacian matrix of $K_n - \{\epsilon_1, \epsilon_2\}$, where $\epsilon_1 = \{1, 2\}$ and $\epsilon_2 = \{3, 4\}$. If n is even, then the spectrum is: eigenvalue 0 of multiplicity 1, with eigenvector $\mathbf{1}$; eigenvalue $n-2$ of multiplicity 2, with eigenvectors a_1 and y_1; and finally eigenvalue n with multiplicity $n-3$, with eigenbasis $\{x_1, x_2, \ldots, x_k, y_2, y_3, \ldots, y_k\}$ with $k = n/2-1$. Similarly, for n odd we have Laplacian spectrum: eigenvalue 0 of multiplicity 1, with eigenvector $\mathbf{1}$; eigenvalue $n-2$ of multiplicity 2, with eigenvectors a_1 and y_1; and finally eigenvalue n with multiplicity $n-2$, with eigenbasis $\{x_1, x_2, \ldots, x_l, y_2, y_3, \ldots, y_l, z\}$ with $l = (n-1)/2 - 1$.

Since we established the quasi-orthogonal nature of these eigenvectors already above, it follows that $K_n - \{\epsilon_1, \epsilon_2\}$ is a weakly Hadamard diagonalizable graph.

More generally, if m is a matching of size s, then without loss of generality we may assume that

$$m = \{\{1, 2\}, \{3, 4\}, \ldots, \{2s-1, 2s\}\}.$$

In this case the Laplacian spectrum for $G - m$ is given by:

- For n even: we need $n \geq 2(s+1)$. Eigenvalue 0 of multiplicity 1, with eigenvector $\mathbf{1}$; eigenvalue $n-2$ of multiplicity s, with eigenbasis $\{a_1, y_1, y_2, \ldots, y_{s-1}\}$; and finally eigenvalue n with multiplicity $n-s-1$, with eigenbasis: $\{x_1, x_2, \ldots, x_k, y_s, \ldots, y_k\}$ with $k = n/2 - 1$.

- For n even: we need $n \geq 2(s+1)+1$. Eigenvalue 0 of multiplicity 1, with eigenvector $\mathbf{1}$; eigenvalue $n-2$ of multiplicity s, with eigenbasis $\{a_1, y_1, y_2, \dots, y_{s-1}\}$; and finally eigenvalue n with multiplicity $n-s-1$, with eigenbasis: $\{x_1, x_2, \dots, x_l, y_s, \dots, y_l, z\}$ with $l = (n-1)/2 - 1$.

As observed above, since we established the quasi-orthogonal nature of these eigenvectors already above, it follows that $K_n - m$ is WHD.

Corollary 4.9. *For $n \geq 4$, a complete graph K_n minus a matching of size s with $s \leq \dfrac{n}{2}$ is WHD.*

5. Strongly-regular graph families

In this section we exhibit some families of strongly-regular graphs that are WHD. All graphs in this section are regular, so it is sufficient to study the eigenbases corresponding to the adjacency matrix, as the Laplacian matrix is simply a translation of the adjacency matrix in this case. It is well-known that the adjacency matrix associated with a strongly-regular graph has exactly three distinct eigenvalues: the degree d, a negative eigenvalue τ, and a positive eigenvalue θ.

The motivation for considering these graphs is that they all have the property that equality holds in the ratio bound (Theorem 5.1 below) and there is a well-known basis of 01-vectors for the span of the d-eigenspace and the τ-eigenspace.

The following is the well-known ratio bound for cocliques, which was originally established by Delsarte (see [9]) and often attributed to Hoffman as well. Further details can be found in [10, Section 2.4]. Recall that a subset S of vertices in a graph is called an independent set (or coclique) if no two vertices in S are adjacent. The size of the largest independent set in a graph G is denoted by $\alpha(G)$.

Theorem 5.1. *Let X be a d-regular graph whose adjacency matrix has the least eigenvalue τ. Then*

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

and, if equality holds for some coclique S with characteristic vector v_S, then

$$v_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is an eigenvector with eigenvalue τ.

There is also a ratio bound for cliques, here we only state the result for strongly-regular graphs, but the bound holds more generally (see [9] or [10, Corollary 3.7.2]).

Theorem 5.2. *Let X be a strongly regular graph with degree d whose adjacency matrix has the least eigenvalue τ. Then*

$$\omega(X) \leq 1 - \frac{d}{\tau}$$

and, if equality holds for some clique C with characteristic vector v_C, then

$$v_C - \frac{|C|}{|V(X)|}\mathbf{1}$$

is an eigenvector with eigenvalue θ.

We consider the following families of strongly regular graphs: Paley graphs, Kneser graphs $K(n,2)$, block graphs of orthogonal arrays and block graphs of designs. For each of these graphs, there is a set of characteristic vectors of some specified vertex sets that span both the d-eigenspace and the τ-eigenspace.

5.1. Paley graphs

Let $\mathbb{F}$ be a finite field of order q with $q \equiv 1 \pmod 4$ and $q = p^2$ for some prime number p. The vertices of the *Paley graph*, are the elements of $\mathbb{F}$, and two vertices are adjacent if and only if their difference is a square in $\mathbb{F}$. The next result is a standard and well-known result on Paley graphs (see [2,6,7] or more recently [10, Section 5.8]).

Theorem 5.3. *Let $P(p^2)$ be a Paley graph with $p^2 \equiv 1 \pmod 4$. Then*

(a) *$P(p^2)$ is self complementary and arc transitive;*
(b) *$P(p^2)$ is a strongly-regular graph with parameters*

$$\left(p^2, \quad (p^2-1)/2; \quad (p^2-5)/4, \quad (p^2-1)/4\right);$$

(c) *The eigenvalues for $P(p^2)$ and respective multiplicities are*

$$\left(\frac{p^2-1}{2}\right)^{(1)}, \left(\frac{p-1}{2}\right)^{\left(\frac{p^2-1}{2}\right)}, \left(-\frac{p+1}{2}\right)^{\left(\frac{p^2-1}{2}\right)}.$$

By Theorem 5.1 and 5.2

$$\alpha(P(p^2)) \leq \frac{p^2}{1 - \frac{p^2-1}{-(1+p)}} = p \text{ and } \omega(P(p^2)) \leq 1 - \frac{p^2-1}{-(1+p)} = p.$$

If $\mathbb{F}$ is a finite field of order p^2 and $\mathbb{E}$ is the sub-field of order p, then the elements of $\mathbb{E}$ induce a clique in $P(p^2)$ of size p. Since a Paley graph $P(p^2)$ is self complementary, we must also have a coclique of size p. This, along with the above bounds, implies that $\alpha(P(p^2)) = p$ and $\omega(P(p^2)) = p$, and equality holds in both Theorem 5.1 and Theorem 5.2.

Let $\mathcal{S}^*$ denote the set of nonzero squares in $\mathbb{F}$. For any a in $\mathcal{S}^*$ and any b in $\mathbb{F}$, the set

$$\mathcal{S}(a, b) = \{ax + b \ : \ x \in \mathbb{E}\}$$

is a clique in $P(p^2)$. These cliques are the square translates of $\mathbb{E}$.

The set $\mathcal{S}^*$ and the set $\mathbb{E}^*$, the non-zero elements in $\mathbb{E}$, are both multiplicative subgroups of $\mathbb{F}$. Further, $\mathbb{E}^* \subset \mathcal{S}^*$, so the set $\mathcal{S}^*/\mathbb{E}^*$ is defined.

Lemma 5.4. *Let $a, a' \in \mathcal{S}^*/\mathbb{E}^*$ and $b, b' \in \mathbb{F}$. Then*

(1) $|\mathcal{S}(a, b) \cap \mathcal{S}(a, b')| = 0$ *if* $b, b' \in \mathbb{F}/a\mathbb{E}$ *with* $b \neq b'$, *and*
(2) $\mathcal{S}(a, b) = \mathcal{S}(a, b')$ *if* $b, b' \in a\mathbb{E}$.

Proof. For Statement 1, assume on the contrary that $\mathcal{S}(a, b) \cap \mathcal{S}(a, b') \neq \emptyset$. Let $y \in \mathcal{S}(a, b) \cap \mathcal{S}(a, b')$. Then y can be written as $y = ax + b = ax' + b'$ for some $x, x' \in \mathbb{E}$. Hence $a(x-x') = b'-b$ or $b-b'$ is a multiple of a which is not possible since $b \neq b' \in \mathbb{F}/a\mathbb{E}$. Therefore, $\mathcal{S}(a, b) \cap \mathcal{S}(a, b') = \emptyset$.

For Statement 2, let $z \in \mathcal{S}(a, b)$. Then $z = ax + b$ for some $x \in \mathbb{E}$. Since $b, b' \in a\mathbb{E}$ we have $b = ae$ and $b' = ae'$ for some $e, e' \in \mathbb{E}$. Whence,

$$z = ax + b = ax + ae + ae' - ae' = a(x + e - e') + ae' = ax' + b',$$

where $x' = x + e - e' \in \mathbb{E}$ which implies that $z \in \mathcal{S}(a, b')$. Therefore, $\mathcal{S}(a, b) \subseteq \mathcal{S}(a, b')$. Similarly, we have $\mathcal{S}(a, b') \subseteq \mathcal{S}(a, b)$ and the result follows. $\square$

Lemma 5.5. *Let $a, a' \in \mathcal{S}^*/\mathbb{E}^*$ with $a \neq a'$. Consider $\mathcal{S}(a, b)$, with $b \in \mathbb{F}/a\mathbb{E}$, and $\mathcal{S}(a', b')$, with $b' \in \mathbb{F}/a'\mathbb{E}$. Then $|\mathcal{S}(a, b) \cap S(a', b')| = 1$.*

Proof. For $a \neq a'$ we have $a\mathbb{E} \cap a'\mathbb{E} = \{0\}$ since for any $\alpha \in a\mathbb{E} \cap a'\mathbb{E}$, we have that $\alpha = ae = a'e'$ for some $e, e' \in \mathbb{E}$. Hence $a = a'e'e^{-1} = a'\beta$, where $\beta = e'e^{-1} \in \mathbb{E}$ which is a contradiction since $a, a' \in \mathcal{S}^*/\mathbb{E}^*$. Therefore, $a\mathbb{E} \cap a'\mathbb{E} = \{0\}$.

Let $z_1, z_2 \in \mathcal{S}(a, b) \cap S(a', b')$ with $z_1 \neq z_2$. Then $z_1 = a\gamma_1 + b = a'\gamma_1' + b'$ and $z_2 = a\gamma_2 + b = a'\gamma_2' + b'$ for some $\gamma_1, \gamma_1', \gamma_2, \gamma_2' \in \mathbb{E}$. Hence $a(\gamma_1 - \gamma_2) = a'(\gamma_1' - \gamma_2')$ and $a(\gamma_1 - \gamma_2), a'(\gamma_1' - \gamma_2') \in a\mathbb{E} \cap a'\mathbb{E}$. Thus $\gamma_1 = \gamma_2$ and $\gamma_1' = \gamma_2'$ which implies that $z_1 = z_2$ and we reach a contradiction. Therefore, $|\mathcal{S}(a, b) \cap S(a', b')| \leq 1$.

Finally, Statement 1 of Lemma 5.4 implies that the sets $\mathcal{S}(a, b)$ with $b \in \mathbb{F}/a\mathbb{E}$ partition the elements of $\mathbb{F}$ into p parts each of size p. Since $| \cup_{b \in \mathbb{F}/a\mathbb{E}} \mathcal{S}(a, b) \cap S(a', b')| = p$, the fact that $|\mathcal{S}(a, b) \cap S(a', b')| \leq 1$ actually implies $|\mathcal{S}(a, b) \cap S(a', b')| = 1$. $\square$

For a fixed $a \in \mathcal{S}^*/\mathbb{E}^*$, define a set of cliques as follows

$$\mathcal{S}_a = \{\mathcal{S}(a,b) \,|\, b \in \mathbb{F}/a\mathbb{E}\}.$$

Lemma 5.6. *The set of characteristic vectors of the cliques in the set $\cup_{a\in\mathcal{S}^*/\mathbb{E}^*}\mathcal{S}_a$ spans the direct sum of the $\frac{p-1}{2}$-eigenspace and* $\mathbf{1}$.

Proof. From Theorem 5.2 we have that the characteristic vectors are in the direct sum of the $\frac{p-1}{2}$-eigenspace and $\mathbf{1}$.

Form a matrix M with the first $|\mathbb{F}/a\mathbb{E}| = p$ columns being the characteristic vectors of the sets $\mathcal{S}_a$ (fix a and vary the values of b). The next p columns are all the characteristic vectors of the cliques in the sets $\mathcal{S}_{a'}$, where $a' \neq a$ is in $\mathcal{S}^*/\mathbb{E}^*$. Now continue in this manner producing such columns for all $\frac{p+1}{2}$ values of $a \in \mathcal{S}^*/\mathbb{E}^*$.

It is clear that the dot product of any two characteristic vectors for two sets from the same $\mathcal{S}_a$ is 0. Similarly, the dot product of two vectors from different sets $\mathcal{S}_a$ is 1. With this we can express the matrix M^tM as follows

$$M^tM = pI_{\frac{p(p+1)}{2}} + \left(\left(J_{\frac{p+1}{2}} - I_{\frac{p+1}{2}}\right) \otimes J_p\right).$$

The spectrum of J_p is $\{p^{(1)}, 0^{(p-1)}\}$. The spectrum of $J_{\frac{p+1}{2}} - I_{\frac{p+1}{2}}$ is $\{\frac{p-1}{2}^{(1)}, -1^{(\frac{p-1}{2})}\}$, and the spectrum of $(J_{\frac{p+1}{2}} - I_{\frac{p+1}{2}}) \otimes J_p$ is

$$\left\{\frac{p(p-1)}{2}^{(1)}, -p^{(\frac{p-1}{2})}, 0^{(\frac{p^2-1}{2})}\right\}.$$

Finally, it follows that the spectrum of M^tM is found by adding p to each of these eigenvalues. So the spectrum of M^tM is

$$\left\{\frac{p(p+1)}{2}^{(1)}, 0^{(\frac{p-1}{2})}, p^{(\frac{p^2-1}{2})}\right\}.$$

The rank of M is equal to the rank M^tM, which is $\frac{p^2+1}{2}$, which is equal to the dimension of the $\frac{p-1}{2}$-eigenspace. $\square$

In a similar fashion, we can apply similar notions in order to exhibit a quasi-orthogonal basis for the $\frac{p-1}{2}$-eigenspace associated with the adjacency matrix of a Paley graph.

Let $a \in \mathcal{S}^*/\mathbb{E}^*$ and order the elements of $\mathbb{F}/a\mathbb{E}$ and label them by $b_1, b_2, \ldots, b_p$. Then $\mathcal{S}(a, b_i)$ is a clique in the Paley graph. Define χ_{a,b_i} to be the characteristic vector of $\mathcal{S}(a, b_i)$ and let

$$\chi_{a,i} = \chi_{a,b_i} - \chi_{a,b_{i+1}}.$$

Then for $a \in \mathcal{S}^*/\mathbb{E}^*$ and $b \in \mathcal{S}^*/\mathbb{E}^*$

$$\chi_{a,i} \cdot \chi_{b,j} = \begin{cases} 2p & \text{if } a = b \text{ and } i = j, \\ -p & \text{if } a = b \text{ and } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

There are $\frac{p^2-1}{2(p-1)} = \frac{p+1}{2}$ choices for an $a \in \mathcal{S}^*/\mathbb{E}^*$, and for each a there are p choices for i. So for each a there are $p - 1$ vectors $\chi_{a,i}$, and in total there are $\frac{p^2-1}{2}$ vectors $\chi_{a,i}$.

Order the elements in $\mathcal{S}^*/\mathbb{E}^*$ and label them by $a_1, a_2, \ldots, a_{\frac{p+1}{2}}$. Define a matrix M to have columns $\chi_{a_k,i}$ and order them so that $\chi_{a_k,i}$ occurs before $\chi_{a_\ell,j}$ whenever $k < \ell$; and whenever $k = \ell$ and $i < j$. Then

$$M^t M = I_{\frac{p+1}{2}} \otimes D,$$

where D is a tridiagonal matrix of size $(p - 1) \times (p - 1)$ with all entries equal to $2p$ on the main diagonal and all entries equal to $-p$ on the super- and sub-diagonals.

The eigenvalues of D are then

$$2p - 2p \cos\left(\frac{k\pi}{p+1}\right), \qquad k = 1, 2, \ldots, p.$$

So D has full rank. It follows that the rank of $M^t M$ is $\frac{p^2-1}{2}$, which implies that the vectors $\chi_{a,i}$ span the $\frac{p-1}{2}$-eigenspace. We restate this result as a lemma.

Lemma 5.7. *Let p be a prime power with $p^2 \equiv 1 \pmod 4$. Then the vectors $\chi_{a,i}$, with entries $\{-1, 0, 1\}$ for $a \in \mathcal{S}^*$ and $i \in \{1, \ldots, p - 1\}$, form a quasi-orthogonal basis of the $\frac{p-1}{2}$-eigenspace of $P(p^2)$.*

Since the Paley graphs are self complementary, we can construct a quasi-orthogonal basis for the $-\frac{p+1}{2}$-eigenspace. Let $\mathcal{N}^*$ denote the set of nonsquares in $\mathbb{F}$. For any a in $\mathcal{N}^*$ and any b in $\mathbb{F}$, the set

$$\mathcal{S}(a, b) = \{ax + b \ : \ x \in \mathbb{E}\}$$

is a coclique in $P(p^2)$. Further, the set of characteristic vectors of cocliques

$$T_a = \{\mathcal{S}(a, b) \,|\, b \in \mathbb{E}\}, \quad a \in \mathcal{N}^*/\mathbb{E}^*$$

spans the direct sum of the θ-eigenspace and $\mathbf{1}$. Again we define χ_{a,b_i} to be the characteristic vector of $\mathcal{S}(a, b_i)$ and let

$$\chi_{a,i} = \chi_{a,b_i} - \chi_{a,b_{i+1}}$$

for any a in $\mathcal{N}^*$. Consequently, we have the following.

Lemma 5.8. *Let p be a prime power with $p^2 \equiv 1 \pmod 4$. Then the $\{-1,0,1\}$-vectors $\chi_{a,i}$ for $a \in \mathcal{N}^*$ form a quasi-orthogonal basis of the $-\frac{p+1}{2}$-eigenspace of $P(p^2)$.*

Putting Lemma 5.7 and 5.8 together we obtain the following fact.

Theorem 5.9. *Let p be a prime power with $p^2 \equiv 1 \pmod 4$. The Paley graph $P(p^2)$ is WHD.*

5.2. Kneser graphs

In this section, we consider the Kneser graph $K(n,2)$ with $n \geq 5$. This graph is strongly regular, and its spectrum is

$$\sigma(K(n,2)) = \left\{ \binom{n-2}{2}^{(1)}, -(n-3)^{(n-1)}, 1^{\left(\binom{n}{2}-n\right)} \right\}.$$

This graph is also isomorphic to the complement of the line graph of a complete graph

Let χ_i be the vector in $\mathbb{R}^{\binom{n}{2}}$ indexed by the 2-subsets of $\{1,2,\dots,n\}$, with the entry corresponding to the set A equal to 1 if $i \in A$ and 0 otherwise. So χ_i is the characteristic vector for the vertices in $K(n,2)$ (so the 2-sets from $\{1,\dots,n\}$) that contain i.

The negative eigenvalue of $K(n,2)$ is $\tau = -(n-3)$. The following follows from Theorem 5.1 and the well-known EKR theorem for intersecting sets.

Proposition 5.10. *The τ-eigenspace is spanned by the vectors $\{\chi_i - \frac{2}{n}\mathbf{1}\}$ for $i = 1,\dots,n$.*

Note that each $\chi_i - \frac{2}{n}\mathbf{1}$ has only two possible entries, namely $-\frac{2}{n}$ or $\frac{n-2}{n}$.

Lemma 5.11. *The vectors $\{\chi_i - \frac{2}{n}\mathbf{1}\}$ for $i = 1,\dots,n$ are the only τ-eigenvectors for $K(n,2)$ that have exactly two different entries (up to scalar multiplication).*

Proof. Let v be a τ-eigenvector with entries from the set $\{x,y\}$. Then, by Proposition 5.10, v is in the span of $\{\chi_i - \frac{2}{n}\mathbf{1} \mid i = 1,\dots,n\}$. That is,

$$v = \sum_{i=1}^{n} a_i(\chi_i - \frac{2}{n}\mathbf{1}).$$

This implies that the linear combination $w = \sum_{i=1}^{n} a_i\chi_i$ is a vector with exactly two distinct entries.

It is easy to see that for the row corresponding to a 2-subset $\{i,j\}$, there are exactly two vectors, namely χ_i and χ_j, with the entry in the row equal to 1; the $\{i,j\}$-entry in all of the other vectors is equal to 0.

Assume there are three distinct coefficients in the equation $w = \sum_{i=1}^{n} a_i\chi_i$. So without loss of generality, assume that a_1, a_2 and a_3 are all distinct. Then the $\{1,2\}$, $\{2,3\}$ and

$\{1,3\}$ entries of w will be, respectively, a_1+a_2, a_2+a_3, a_1+a_3. Since these values are all distinct, w will have at least three distinct entries. So the linear combination can have at most two distinct coefficients.

Next assume that $a_1 = a_2 = a$ and $a_3 = a_4 = b$, where $a \neq b$, then w will have the three distinct numbers $a+b$, $2a$, $2b$ in its $\{1,3\}, \{1,2\}, \{3,4\}$ entries. So one of the two distinct coefficients in the linear combination can occur only once.

Thus the linear combination must have all but one coefficient equal. Finally, since v is a τ-eigenvector, it is also orthogonal to the all ones vector, these two facts together imply that v must be a scalar multiple of a $\chi_i - \frac{2}{n}\mathbf{1}$. □

Corollary 5.12. *The τ-eigenspace of $K(n,2)$ does not have an orthogonal basis of vectors whose entries take only two values.*

Proof. By the previous result, any τ-eigenvector that takes only two values must be a scalar multiple of some $\chi_i - \frac{2}{n}$. But for distinct $i, j \in [n-1]$

$$\left(\chi_i - \frac{2}{n}\mathbf{1}\right)^t \left(\chi_j - \frac{2}{n}\mathbf{1}\right) = 1 - \frac{2(n-1)}{n} < 0. \quad \square$$

The above argument shows that $K(n,2)$ is not a Hadamard diagonalizable graph for any n. However, we do not know whether it is a weakly Hadamard diagonalizable graph or not. We can prove that there is a quasi-orthogonal basis of τ-eigenvectors with entries from $\{0,-1,1\}$.

Proposition 5.13. *The vectors $\{\chi_i - \chi_{i+1} | 1 \leq i \leq n-1\}$ form a quasi-orthogonal basis for the τ-eigenspace of $K(n,2)$.*

Proof. It follows from Theorem 5.2 that $\chi_i - \chi_{i+1}$ are τ-eigenvectors.

Let M be the $(n-1)\times(n-1)$ matrix with the vectors $\chi_i - \chi_{i+1}$ as the columns. Then M^tM is a tridiagonal matrix with main diagonal entries all equal to $2n-4$, and all of the off-diagonal entries are equal to $-(n-2)$.

The eigenvalues of M^tM are

$$(2n-4) - 2(n-2)\cos\left(\frac{k\pi}{n}\right), \qquad k = 1, 2, \ldots, n-1$$

so it has full rank and is a basis for the τ-eigenspace. □

We end this section with this question: Are the graphs $K(n,2)$ WHD? To solve this we would need to find a quasi-orthogonal basis for the θ-eigenspace. For $n = 5, 6$, we provide a positive resolution to this question.

Lemma 5.14. *The graphs $K(5,2)$ and $K(6,2)$ are both WHD.*

Proof. The rows of the matrix below form a quasi-orthogonal basis for $K(5,2)$ (the first vector is for the eigenvalue 3, the next four are -2-eigenvectors and the final five are 1-eigenvectors)

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

The rows of the matrix P' below form a quasi-orthogonal basis for $K(6,2)$ (the first vector is for the eigenvalue 6, the next five are -3-eigenvectors and the final nine are 1-eigenvectors.)

$$P' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}. \quad \square$$

5.3. *Orthogonal array graphs*

An $m \times n^2$ array with entries from $\{1, 2, \ldots, n\}$ is called an *orthogonal array*, denoted by $OA(m,n)$, if the columns of any $2 \times n^2$ subarray consist of all n^2 ordered pairs of elements from $\{1, 2, \ldots, n\}$. In particular, for any two rows each ordered pair from $\{1, 2, \ldots, n\}$ occurs in exactly one column. The *block graph for an orthogonal array* $OA(m,n)$ is a strongly-regular graph defined from an orthogonal array. The columns of the array are the vertices of the graph, and two vertices are adjacent if there is a row in which the two columns have the same entry. This graph is denoted by $X_{OA(m,n)}$.

It is well-known that for any orthogonal array $OA(m,n)$ we have $m \leq n+1$ (see for example [8, Section III.3]). Further, if $m = n+1$, then $X_{OA(n+1,n)}$ is the complete graph on n^2 vertices. The eigenvalues of $X_{OA(m,n)}$ for any $OA(m,n)$ are well-known (see, for example, [10, Section 5.5]).

Theorem 5.15. *If $OA(m,n)$ is an orthogonal array where $m < n+1$, then its block graph $X_{OA(m,n)}$ is strongly regular, with spectrum (for the adjacency matrix)*

$$\left\{m(n-1)^{(1)}, \quad (n-m)^{(m(n-1))}, \quad -m^{((n-1)(n+1-m))}\right\}.$$

We can apply Theorem 5.2 to $X_{OA(m,n)}$ to deduce

$$\omega(X_{OA(m,n)}) \leq 1 - \frac{m(n-1)}{-m} = n.$$

The set of columns of $OA(m,n)$ that have the same entry in the same row form a clique in $X_{OA(m,n)}$ that meets this bound. For $i \in \{1,\dots,n\}$ let $S_{r,i}$ denote the set of columns of $OA(m,n)$ that have the entry i in row r. Further, define $v_{r,i}$ to be the characteristic vector for $S_{r,i}$.

Theorem 5.16. *Let $OA(m,n)$ be an orthogonal array with $m < n+1$. The set of vectors*

$$\{v_{r,i} - v_{r,i+1} \,|\, r \in \{1,\dots,m\}, i \in \{1,\dots,n-1\}\}$$

is a quasi-orthogonal basis for the $(n-m)$-eigenspace of $X_{OA(m,n)}$.

Proof. From Theorem 5.2 $v_{r,i} - \frac{1}{n}\mathbf{1}$ is a $(n-m)$-eigenvector, so $v_{r,i} - v_{r,i+1}$ is also a $(n-m)$-eigenvector for $r \in \{1,\dots,m\}$ and $i \in \{1,\dots,n-1\}$.

Define H_r to be the $(n-1)\times(n-1)$ matrix with columns $v_{r,i}-v_{r,i+1}$ for $i = 1,\dots,n-1$. Then $H_r^t H_r$ is tridiagonal with all entries equal to $2n$ on the main diagonal and all entries equal to $-n$ on the super- and sub-diagonal. Then $H_r^t H_r$ has full rank since the eigenvalues are

$$2n - 2n\cos\left(\frac{k\pi}{n}\right), \qquad k = 1, 2, \dots, n-1.$$

Define $H = [H_1|H_2|\dots|H_m]$. Since for any $r \neq s$, and any $i, j \in \{1,\dots,n\}$ we have $(v_{r,i} - v_{r,i+1}) \cdot (v_{s,j} - v_{s,j+1}) = 0$, it follows that $H^t H = \bigoplus_{r=1}^m H_r^t H_r$. Thus $H^t H$ has full rank (equal to $m(n-1)$) and these vectors span the $(n-m)$-eigenspace of $X_{OA(m,n)}$. $\square$

It is unclear at this time if there exists a quasi-orthogonal basis for the $(-m)$-eigenspace. Consequently, we pose the following question.

Question 5.1. Is there a quasi-orthogonal basis for the $-m$-eigenspace, or more specifically, is the graph $X_{OA(m,n)}$ WHD?

MacNeish's construction (see [13,15] or more recently [10, Section 5.5]) can be used to build an $OA(m, n^2)$ from an $OA(m, n)$. If the columns of the $OA(m, n)$ are denoted by c_i, then the columns of the $OA(m, n^2)$ are given by $c_i + nc_j$ for $i, j \in \{1, \ldots, n^2\}$.

For example, the following $OA(3, 2)$

$$OA_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

can be used to construct this $OA(3, 4)$

$$OA_2 = \begin{bmatrix} 0011 & 0011 & 2233 & 2233 \\ 0101 & 2323 & 0101 & 2323 \\ 0110 & 2332 & 2332 & 0110 \end{bmatrix}.$$

Denote the columns of an $OA(m, n)$ by $c_i, c_2, \ldots, c_{n^2}$. For each row in the orthogonal array, define an $n^2 \times n^2$ matrix M_i with $i \in \{1, \ldots, m\}$; we call these the *row matrices of* $OA(m, n)$. The rows and columns of M_i are indexed by the columns in $OA(m, n)$ and the (c_i, c_j)-entry of M_i is 1 if c_i and c_j agree in row i, and zero otherwise. Then $X_{OA(m,n)} = \sum_{i=1}^{m}(M_i - I)$.

Lemma 5.17. *Let $OA(m, n)$ be an orthogonal array with row matrices M_i. If $OA(m, n^2)$ is the orthogonal array formed from McNeish's construction with $OA(m, n)$, then the row matrices of $OA(m, n^2)$ are $M_i \otimes M_i$.*

Proof. Let c_i be the columns of the $OA(m, n)$. Then the columns of $OA(m, n^2)$ are $c_i + nc_j$. So columns $c_i + nc_j$ and $c_k + nc_\ell$ intersect in row r if and only if both c_i and c_k, and c_j and c_ℓ intersect in row r. So the rth row matrix for $OA(m, n^2)$ is $M_r \otimes M_r$. □

Starting with the orthogonal array OA_1 defined above, recursively define OA_k to be the $OA(m, 2^{2^k})$ formed by MacNeish's construction on OA_{k-1}.

Lemma 5.18. *For all k, the graph X_{OA_k} is Hadamard diagonalizable.*

Proof. The row matrices for OA_1 are Hadamard diagonalizable by the Sylvester Hadamard matrix H_4. So the row matrices of OA_k are Hadamard diagonalizable by the Sylvester Hadamard matrix $H_{2^{2^k}}$. Thus the matrix X_{OA_k} is Hadamard diagonalizable by $H_{2^{2^k}}$. □

This result also follows from the fact that the graph X_{OA_k} is cubelike (this means that it is a Cayley graph for the group $\mathbb{Z}_2^d$) and in [5] it is noted that any cubelike graph is diagonalized by the Sylvester Hadamard matrix.

We can also find a large family of block graphs of orthogonal arrays that are WHD.

Theorem 5.19. *Let $O = OA(m,n)$ be an orthogonal array that can be extended to an orthogonal array with $n+1$ rows. Then $X_{OA(m,n)}$ is WHD.*

Proof. Let O' be the orthogonal array with $n+1$ rows that is an extension of O. For each row that is in O', but not in O, define n vectors $v_1, v_2, \dots, v_n$ each of length n^2. The ith entry of v_i is equal to 1 if the ith entry of the row is equal to i and 0 otherwise. Then the vectors $v_i - v_{i+1}$ are $(-m)$-eigenvectors with entries from $\{0, -1, 1\}$. To see this, let r be a row in O' that is not in O. Consider the set of columns in O, in which row r contains element i. No two of these columns can have the same entry in the same row; if they did, then there will be a repeated pair of elements in this row and r in O'. This means that v_i is the characteristic vector of a clique in X_O. It follows from Theorem 5.2 that $v_i - v_{i+1}$ is an $(-m)$-eigenvector.

Since this is done for each row of O' that is not in O, we produce $(n+1-m)(n-1)$ vectors. Since these vectors come from rows of an orthogonal array they have the quasi-orthogonal property. □

An orthogonal array with $n+1$ rows is the largest possible orthogonal array and its block graph is a complete graph, which is WHD. At the opposite end of the spectrum is an orthogonal array with only two rows. In this case, $X_{OA(2,n)} = K_n \,\square\, K_n$, but it is still open if this graph is WHD for all n.

If an $n \times n$ Hadamard matrix exists, then $K_n \,\square\, K_n$ is Hadamard diagonalizable, and hence WHD. Further, $K_3 \,\square\, K_3$ is WHD. This is graph number 3 on nine vertices in the Appendix. We conjecture that all of these graphs are WHD.

Conjecture 5.1. *For all positive integers n the graph $K_n \,\square\, K_n$ is WHD.*

So the next question is when is the orthogonal array graph associated with an orthogonal array with only three rows is WHD? Such an orthogonal array is equivalent to a Latin square. Observe that each column of such an array has three letters, and each column describes an entry in a Latin square; the first two letters give the row and the column and the third letter is the entry is the given row and column.

Question 5.2. When is a Latin square graph WHD?

5.4. Block graph for a 2-$(n,m,1)$ design

Assume that $(V, \mathcal{B})$ is a 2-$(n,m,1)$ design that is not symmetric. The *block graph of the* 2-$(n,m,1)$ *design* $(V,\mathcal{B})$ is the graph with the blocks of the design as the vertices in which two blocks are adjacent if and only if they intersect. It is well-known that this graph is a strongly-regular graph (see, for example, [4,14,16] or more recently [10, Section 5.3]).

Theorem 5.20. *The block graph of a 2-$(n,m,1)$ design (that is not symmetric) is strongly regular with spectrum (for the adjacency matrix)*

$$\left\{ \frac{m(n-m)}{m-1}^{(1)}, \frac{n-m^2}{m-1}^{(n-1)}, -m^{\left(\frac{n(n-1)}{m(m-1)}-n\right)} \right\}.$$

By Theorem 5.2,

$$\omega(X_{(V,\mathcal{B})}) \leq 1 - \frac{k}{\tau} = 1 - \frac{\frac{m(n-m)}{(m-1)}}{-m} = \frac{n-1}{m-1}.$$

This number is r, the *replication number* for the design, this is the number of blocks that contain a given $i \in \{1, \dots, n\}$. For any $i \in \{1, \dots, n\}$ let S_i be the collection of all blocks in the design that contain i, which forms a clique of size r. The cliques S_i are called the *canonical cliques* of the block graph. Let v_i be the characteristic vectors of the canonical clique S_i.

Lemma 5.21. *Let $(V, \mathcal{B})$ be any 2-$(n, m, 1)$ design. Then the set $\{v_i - v_{i+1} \mid i \in \{1, \dots, n\}\}$ is a quasi-orthogonal basis for the $(\frac{n-m^2}{m-1})$-eigenspace of block graph of $(V, \mathcal{B})$.*

Proof. It follows from Theorem 5.2 that the vectors $v_i - v_{i+1}$ are $(\frac{n-m^2}{m-1})$-eigenvectors.

Let H be the matrix whose columns are $v_i - v_{i+1}$. Then H^tH is tridiagonal, with all entries equal to $2r - 2$ on the main diagonal and all entries equal to $r - 1$ on the super- and sub-diagonal. As in the previous examples, this matrix has full rank (specifically, the rank is $n - 1$). □

It is still open if the τ-eigenspace has a quasi-orthogonal basis with all entries in $\{0, 1, -1\}$.

Question 5.3. Is there a quasi-orthogonal basis for the $(-m)$-eigenspace with entries from $\{0, -1, 1\}$?

6. Further work

This is a first paper considering WHD graphs, so there are still many open questions. We conclude with two families of graphs which we believe would be interesting to determine if they have the property of being WHD or not.

For $n \geq 1$ the unitary Cayley graph U_n is the Cayley graph of the group $\mathbb{Z}_n$ with connection set the set of all elements that are invertible under multiplication. Unitary Cayley graphs are integral graphs; that is, their eigenvalues are integers.

Question 6.1. For which n is U_n WHD?

Cographs are formed by taking isolated vertices with operations joins and unions. It is known that there is a basis for the Laplacian matrix of a cograph that contains vectors that each only have two entries.

Question 6.2. Which cographs are WHD?

Declaration of competing interest

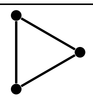
None declared.

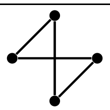
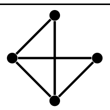
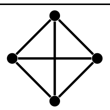
Acknowledgements

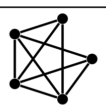
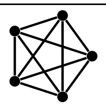
The work in this paper was a joint project of the Discrete Mathematics Research Group at the University of Regina, attended by all of the authors. Dr. Fallat's research was supported in part by NSERC Discovery Research Grant, Application No.: RGPIN-2019-03934. Dr. Meagher's research was supported in part by an NSERC Discovery Research Grant, Application No.: RGPIN-03952-2018. Dr. Nasserasr's research was supported in part by an NSERC Discovery Research Grant, Application No.: RGPIN-2019-05275.

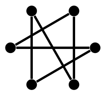
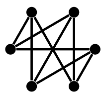
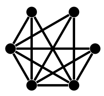
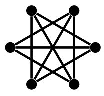
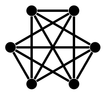
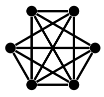
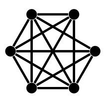
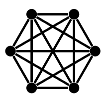
Appendix A. WHD graphs on at most nine vertices

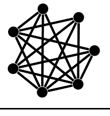
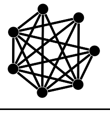
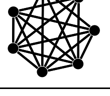
In this Appendix we list all the connected graphs on at most nine vertices that are WHD. Beginning with graphs on 3 vertices, each row of each table consists of a graph along with its corresponding Laplacian spectrum. Further, for each eigenvalue a corresponding quasi-orthogonal basis is exhibited for that eigenspace.

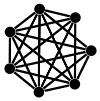
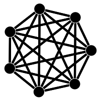
1		0: [1, 1, 1] 3: [1, -1, 0], [0, 1, -1]

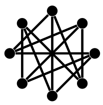
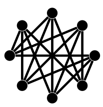
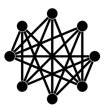
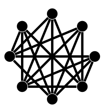
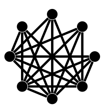
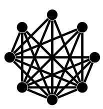
1		4: [1, 1, -1, -1] 0: [1, 1, 1, 1] 2: [1, -1, 0, 0], [0, 0, 1, -1]
2		2: [1, -1, 0, 0] 0: [1, 1, 1, 1] 4: [1, 1, -1, -1], [0, 0, 1, -1]
3		0: [1, 1, 1, 1] 4: [1, 0, 0, -1], [0, 1, 0, -1], [0, 0, 1, -1]

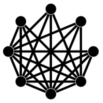
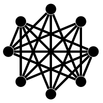
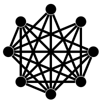
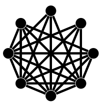
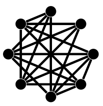
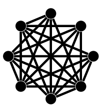
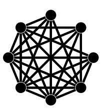
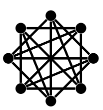
1		3: [1, -1, 0, 0, 0] 0: [1, 1, 1, 1, 1] 5: [1, 1, 0, -1, -1], [0, 0, 1, 0, -1], [0, 0, 0, 1, -1]
2		0: [1, 1, 1, 1, 1] 5: [1, -1, 0, 0, 0], [0, 1, -1, 0, 0], [0, 0, 1, -1, 0], [0, 0, 0, 1, -1]

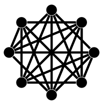
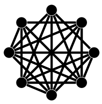
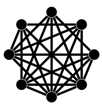
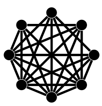
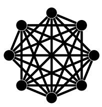
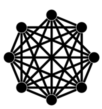
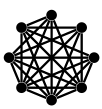
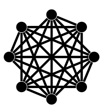
1		4: 0: 3: 1:	[1, 1, 1, -1, -1, -1] [1, 1, 1, 1, 1, 1] [1, 0, -1, -1, 0, 1], [0, 1, -1, -1, 1, 0] [1, 0, -1, 1, 0, -1], [0, 1, -1, 1, -1, 0]
2		6: 0: 3:	[1, 1, 1, -1, -1, -1] [1, 1, 1, 1, 1, 1] [1, -1, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0], [0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 1, -1]
3		0: 3: 6:	[1, 1, 1, 1, 1, 1] [1, 0, -1, 0, 0, 0], [0, 1, -1, 0, 0, 0] [1, 1, 1, -1, -1, -1], [0, 0, 0, 1, 0, -1], [0, 0, 0, 0, 1, -1]
4		2: 0: 5: 3:	[1, -1, 1, -1, 1, -1] [1, 1, 1, 1, 1, 1] [1, 0, -1, -1, 0, 1], [0, 1, 1, 0, -1, -1] [1, 0, -1, 1, 0, -1], [0, 1, -1, 0, 1, -1]
5		0: 6: 4:	[1, 1, 1, 1, 1, 1] [1, 1, 0, 0, -1, -1], [0, 0, 1, 1, -1, -1] [1, -1, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 1, -1]
6		0: 4: 6:	[1, 1, 1, 1, 1, 1] [1, -1, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0] [1, 1, 0, 0, -1, -1], [0, 0, 1, 1, -1, -1], [0, 0, 0, 0, 1, -1]
7		4: 0: 6:	[1, -1, 0, 0, 0, 0] [1, 1, 1, 1, 1, 1] [1, 1, -1, -1, 0, 0], [0, 0, 1, 1, -1, -1], [0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 1, -1]
8		0: 6:	[1, 1, 1, 1, 1, 1] [1, -1, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0], [0, 0, 1, -1, 0, 0], [0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 1, -1]

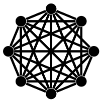
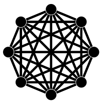
1		5: 0: 4: 7:	[0, 0, 0, 1, -1, 0, 0] [1, 1, 1, 1, 1, 1, 1] [1, 0, -1, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0] [0, 0, 0, 1, 1, -1, -1], [1, 1, 1, -1, -1, 0, -1], [0, 0, 0, 0, 0, 1, -1]
2		0: 4: 7:	[1, 1, 1, 1, 1, 1, 1] [1, 0, -1, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0] [0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 1, -1], [1, 1, 1, -1, -1, -1, 0]
3		0: 5: 7:	[1, 1, 1, 1, 1, 1, 1] [1, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0] [-1, -1, 1, 1, 0, 0, 0], [1, 1, 0, 0, -1, 0, -1], [0, 0, 0, 0, 1, 0, -1], [0, 0, 0, 0, 0, 1, -1]

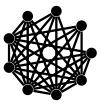
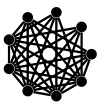
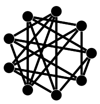
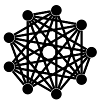
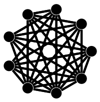
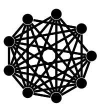
4		5: [1, -1, 0, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1] 7: [0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 1, -1], [0, 0, 1, 1, -1, -1, 0], [1, 1, -1, -1, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0]
5		0: [1, 1, 1, 1, 1, 1, 1] 7: [1, -1, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 1, -1]

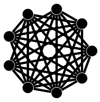
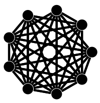
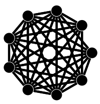
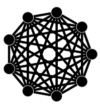
1		6: [1, 1, 1, 1, -1, -1, -1, -1] 0: [1, 1, 1, 1, 1, 1, 1, 1] 4: [1, 0, 0, -1, -1, 0, 0, 1], [0, 1, 0, -1, -1, 0, 1, 0], [0, 0, 1, -1, -1, 1, 0, 0] 2: [1, 0, 0, -1, 1, 0, 0, -1], [0, 1, 0, -1, 1, 0, -1, 0], [0, 0, 1, -1, 1, -1, 0, 0]
2		8: [1, 1, 1, 1, -1, -1, -1, -1] 0: [1, 1, 1, 1, 1, 1, 1, 1] 4: [1, -1, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 1, -1]
3		8: [1, 1, 1, 1, -1, -1, -1, -1] 6: [0, 0, 0, 0, 1, 0, 0, -1] 0: [1, 1, 1, 1, 1, 1, 1, 1] 4: [1, -1, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, -1, 1], [0, 0, 0, 0, 0, 1, -1, 0]
4		8: [1, 1, 1, 1, -1, -1, -1, -1] 0: [1, 1, 1, 1, 1, 1, 1, 1] 6: [0, 0, 0, 0, 1, 0, -1, 0], [0, 0, 0, 0, 0, 1, 0, -1] 4: [1, -1, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 1, -1]
5		0: [1, 1, 1, 1, 1, 1, 1, 1] 8: [1, 1, 1, 1, -1, -1, -1, -1], [0, 0, 0, 0, 1, 1, -1, -1] 6: [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1] 4: [1, -1, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0]
6		6: [0, 0, 0, 0, 1, -1, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1] 8: [1, 1, 1, 1, -1, -1, -1, -1], [0, 0, 0, 0, 1, 1, -1, -1], [0, 0, 0, 0, 0, 0, 1, -1] 4: [1, -1, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0]

7		0: [1, 1, 1, 1, 1, 1, 1, 1] 4: [1, -1, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0] 8: [1, 1, 1, 1, -1, -1, -1, -1], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 1, -1]
8		8: [1, 1, 1, 1, -1, -1, -1, -1] 0: [1, 1, 1, 1, 1, 1, 1, 1] 6: [1, 0, 0, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, -1] 4: [1, -1, -1, 1, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, -1, 1], [0, 0, 0, 0, 0, 1, -1, 0]
9		0: [1, 1, 1, 1, 1, 1, 1, 1] 8: [1, 1, 1, 1, -1, -1, -1, -1], [0, 0, 0, 0, 1, 1, -1, -1] 4: [1, -1, -1, 1, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0] 6: [1, 0, 0, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1]
10		6: [1, 0, 0, -1, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1] 4: [1, -1, -1, 1, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0] 8: [1, 1, 1, 1, -1, -1, -1, -1], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 1, -1]
11		8: [1, 1, 1, -1, -1, -1, -1, 1] 0: [1, 1, 1, 1, 1, 1, 1, 1] 6: [1, 0, 0, 0, 0, 0, 0, -1], [0, 0, 0, 1, 0, -1, 0, 0], [0, 0, 0, 0, 1, 0, -1, 0] 4: [1, -1, -1, 0, 0, 0, 0, 1], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 0, 1, -1, 1, -1, 0]
12		0: [1, 1, 1, 1, 1, 1, 1, 1] 6: [0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0] 5: [1, 0, -1, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0] 8: [0, 0, 0, 1, 1, -1, -1, 0], [1, 1, 1, -1, -1, 0, 0, -1], [-1, -1, -1, 1, 1, 1, 1, -1]
13		0: [1, 1, 1, 1, 1, 1, 1, 1] 5: [1, 0, -1, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0] 8: [0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 1, -1], [1, 1, 1, -1, -1, -1, -1, 1]
14		2: [1, -1, 1, -1, 1, -1, 1, -1] 0: [1, 1, 1, 1, 1, 1, 1, 1] 6: [1, 0, 0, 1, 0, -1, -1, 0], [0, 0, 1, 1, 0, 0, -1, -1], [0, 1, 1, 0, -1, 0, 0, -1] 4: [1, 0, 0, -1, 0, 1, -1, 0], [0, 0, 1, -1, 0, 0, -1, 1], [0, 1, -1, 0, 1, 0, 0, -1]

15		8: [1, 1, 1, 1, -1, -1, -1, -1] 0: [1, 1, 1, 1, 1, 1, 1, 1] 4: [1, -1, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 1, -1] 6: [1, 0, -1, 0, 0, 0, 0, 0], [0, 1, 0, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, -1, 0],[0, 0, 0, 0, 0, 1, 0, -1]
16		4: [1, -1, 1, -1, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1] 8: [1, 1, 1, 1, -1, -1, -1, -1], [0, 0, 0, 0, 1, 1, -1, -1] 6: [1, 0, -1, 0, 0, 0, 0, 0], [0, 1, 0, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1]
17		4: [1, -1, 1, -1, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1] 8: [1, 1, 1, 1, -1, -1, -1, -1], [0, 0, 0, 0, 1, 1, -1, -1], [0, 0, 0, 0, 0, 0, 1, -1] 6: [1, 0, -1, 0, 0, 0, 0, 0], [0, 1, 0, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0]
18		4: [1, -1, 1, -1, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1] 6: [1, 0, -1, 0, 0, 0, 0, 0], [0, 1, 0, -1, 0, 0, 0, 0] 8: [1, 1, 1, 1, -1, -1, -1, -1], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 1, -1]
19		0: [1, 1, 1, 1, 1, 1, 1, 1] 6: [1, -1, 0, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0] 8: [1, 1, -1, -1, 0, 0, 0, 0], [0, 0, 1, 1, -1, -1, 0, 0], [0, 0, 0, 0, 1, 1, -1, -1], [0, 0, 0, 0, 0, 0, 1, -1]
20		0: [1, 1, 1, 1, 1, 1, 1, 1] 6: [1, -1, 0, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0] 8: [1, 1, -1, -1, 0, 0, 0, 0], [0, 0, 1, 1, 1, -1, -1, -1], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 1, -1]
21		6: [0, 0, 0, 1, -1, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1] 5: [1, 0, -1, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0] 8: [0, 0, 0, 0, 0, 0, 1, -1], [0, 0, 0, 0, 0, 1, -1, 0], [1, 1, 1, -1, -1, 1, -1, -1], [1, 1, 1, 0, 0, -1, -1, -1]
22		0: [1, 1, 1, 1, 1, 1, 1, 1] 8: [1, 1, 0, 0, 0, 0, -1, -1], [1, 1, 0, 0, -1, -1, 0, 0], [1, 1, -1, -1, -1, -1, 1, 1] 6: [1, -1, 0, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1]

23		6: [1, -1, 0, 0, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1] 8: [1, 1, -1, -1, 0, 0, 0, 0], [0, 0, 1, 1, -1, -1, 0, 0], [0, 0, 0, 0, 1, 1, -1, -1], [0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1]
24		0: [1, 1, 1, 1, 1, 1, 1, 1] 8: [1, -1, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 1, -1]

1		0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 5: [1, 0, 0, -1, 0, 0, 0, 0, 0], [0, 1, 0, -1, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0, 0] 9: [0, 0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 0, 1, -1], [1, 1, 1, 1, -1, -1, -1, -1, 0]
2		7: [1, 0, 0, -1, 0, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 5: [1, -1, -1, 1, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0, 0] 9: [0, 0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 0, 1, -1], [1, 1, 1, 1, -1, -1, -1, -1, 0]
3		0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 6: [0, 0, 0, 1, 1, 1, -1, -1, -1], [0, 0, 1, 0, -1, -1, 0, 0, 1], [0, 1, -1, -1, 1, 0, 0, 1, -1], [1, -1, 0, 0, -1, 1, 1, -1, 0] 3: [0, 1, -1, 0, -1, 1, -1, 0, 1], [0, 0, 0, 0, 1, -1, 1, -1, 0], [0, 0, 0, 1, -1, 0, 0, 1, -1], [1, -1, 0, -1, 1, 0, -1, 0, 1]
4		0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 9: [1, 1, 1, 0, 0, 0, -1, -1, -1], [0, 0, 0, 1, 1, 1, -1, -1, -1] 6: [1, -1, 0, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 0, 1, -1]
5		0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 9: [1, 1, 1, 0, 0, 0, -1, -1, -1], [0, 0, 0, 1, 1, 1, -1, -1, -1], [0, 0, 0, 0, 0, 0, 1, 0, -1], [0, 0, 0, 0, 0, 0, 0, 1, -1] 6: [1, -1, 0, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0, 0]
6		5: [1, -1, 1, -1, 0, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 7: [1, 0, -1, 0, 0, 0, 0, 0, 0], [0, 1, 0, -1, 0, 0, 0, 0, 0] 9: [0, 0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 0, 1, -1], [1, 1, 1, 1, -1, -1, -1, -1, 0]

7		0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 7: [1, -1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0, 0] 9: [0, 0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 0, 1, -1], [0, 0, 0, 0, 1, 1, -1, -1, 0], [0, 0, 1, 1, -1, -1, 0, 0, 0], [1, 1, -1, -1, 0, 0, 0, 0, 0]
8		0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 7: [1, -1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0, 0] 9: [0, 0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 0, 1, -1], [0, 0, 0, 0, 1, 1, -1, -1, 0], [0, 0, 1, 1, -1, -1, 0, 0, 0], [1, 1, -1, -1, 0, 0, 0, 0, 0]
9		7: [1, -1, 0, 0, 0, 0, 0, 0, 0] 0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 9: [0, 0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0, 0], [1, 1, -1, -1, 0, 0, 0, 0, 0], [0, 0, 1, 1, -1, -1, 0, 0, 0], [0, 0, 0, 0, 1, 1, -1, -1, 0], [0, 0, 0, 0, 0, 0, 0, 1, -1], [0, 0, 0, 0, 0, 0, 1, -1, 0]
10		0: [1, 1, 1, 1, 1, 1, 1, 1, 1] 9: [1, -1, 0, 0, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 0, 0, 0, 0], [0, 0, 1, -1, 0, 0, 0, 0, 0], [0, 0, 0, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 0, 0, 1, -1, 0], [0, 0, 0, 0, 0, 0, 0, 1, -1]

References

[1] S. Akbari, A. Alipour, E. Ghorbani, G.B. Khosrovshahi, $\{-1, 0, 1\}$-basis for the null space of a forest, Linear Algebra Appl. 414 (2006) 505–511.

[2] R.D. Baker, G.L. Ebert, J. Hemmeter, A.J. Woldar, Maximal cliques in the Paley graph of square order, J. Stat. Plan. Inference 56 (1996) 33–38.

[3] S. Barik, S. Fallat, S. Kirkland, On Hadamard diagonalizable graphs, Linear Algebra Appl. 435 (2011) 1885–1902.

[4] R.C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, Pac. J. Math. 18 (1963) 389–419.

[5] A. Bernasconi, C. Godsil, S. Severini, Quantum networks on cubelike graphs, Phys. Rev. A 78 (2008) 052320.

[6] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

[7] S.D. Cohen, Clique numbers of Paley graphs, Quaest. Math. 11 (2) (1988) 225–231.

[8] C.J. Colbourn, J.H. Dinitz (Eds.), Handbook of Combinatorial Designs, second edition, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[9] P. Delsarte, An Algebraic Approach to the Association Schemes of Coding Theory, Philips Res. Rep. Suppl., vol. 10, 1973, vi+97 pp.

[10] C. Godsil, K. Meagher, Erdős-Ko-Rado Theorems: Algebraic Approaches, Cambridge University Press, Cambridge, UK, 2015.

[11] N. Johnston, S. Kirkland, S. Plosker, R. Storey, X. Zhang, Perfect quantum state transfer using Hadamard diagonalizable graphs, Linear Algebra Appl. 534 (2017) 375–398.

[12] H. Kharaghani, S. Suda, Balancedly splittable Hadamard matrices, Discrete Math. 342 (2019) 546–561.

[13] H.F. MacNeish, Euler squares, Ann. Math. 23 (1921–1922) 221–227.
[14] A. Neumaier, Strongly regular graphs with smallest eigenvalue $-m$, Arch. Math. 33 (1979) 392–400.
[15] D.R. Stinson, Combinatorial Designs, Springer-Verlag, New York, 2004.
[16] D.M. Thompson, Eigengraphs: constructing strongly regular graphs with block designs, Util. Math. 20 (1981) 83–115.