ON THE GROUP ASSOCIATION SCHEME OF THE MÖBIUS GROUP

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Abstract. We study properties of the group association scheme of the Möbius group $\mathrm{PGL}_2(q)$, where q is a prime power. First, we determine the automorphism group of the scheme by finding the automorphism group of the derangement graph of the permutation group $\mathrm{PGL}_2(q)$ acting on the projective line $\mathrm{PG}_1(q)$. In particular, we show that the automorphism group of the scheme is generated by the left-regular representation of $\mathrm{PGL}_2(q)$, the right-regular representation of $\mathrm{PGL}_2(q)$, and the anti-automorphism of $\mathrm{PGL}_2(q)$ mapping an element to its inverse. Then, we use the automorphism group to show that the scheme is triply transitive if and only if $q = 2$.

1. Introduction

Let Ω be a finite nonempty set. An ***association scheme*** on Ω is a pair $(\Omega, \mathscr{R})$ where $\mathscr{R} = \{R_0, R_1, \ldots, R_d\}$ is a partition of $\Omega \times \Omega$ with the properties:

(AS1) $R_0 = \{(\omega, \omega) : \omega \in \Omega\}$,

(AS2) for any $0 \leqslant i \leqslant d$, the relation $R_i^* = \{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$ belongs to $\mathscr{R}$,

(AS3) for any $0 \leqslant i, j, k \leqslant d$, there exists a non-negative integer p_{ij}^k such that if $(\alpha, \beta) \in R_k$, then

$$|\{\omega \in \Omega : (\alpha, \omega) \in R_i \text{ and } (\omega, \beta) \in R_j\}| = p_{ij}^k.$$

A pair $(\Omega, \mathscr{R})$ satisfying (AS1)–(AS3) with $|\mathscr{R}| = d + 1$ will also be referred as a d-class association scheme. It is worth noting that there is no consensus on the definition of what an association scheme is, and the standard textbooks (see [2]) on this topic all have slightly different ways of defining these objects. The association schemes in our definition, i.e. those satisfying in (AS1)–(AS3), are also known as homogeneous coherent configurations. If in addition we have $p_{ij}^k = p_{ji}^k$ for all $0 \leqslant i, j, k \leqslant d$, then we say that the association scheme $(\Omega, \mathscr{R})$ is ***commutative***. If $R_i^* = R_i$ for any $0 \leqslant i \leqslant d$, then $(\Omega, \mathscr{R})$ is a ***symmetric*** association scheme.

Given a d-class association scheme $(\Omega, \mathscr{R} = \{R_i\}_{i=0}^d)$, for any $1 \leqslant i \leqslant d$, we let Γ_i be the digraph whose vertex set is Ω, and whose set of arcs is R_i. Note that, for $1 \leqslant i \leqslant d$, $R_i^* = R_i$ if and only if Γ_i is an undirected graph. The adjacency matrix of the digraph Γ_i is denoted by A_i for $1 \leqslant i \leqslant d$. Moreover, we let A_0 be the diagonal matrix in $\mathrm{M}_\Omega(\mathbb{C})$. A

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combinatorial automorphism of $(\Omega, \mathscr{R})$ is a permutation of Ω preserving each relation in $\mathscr{R}$, that is, an element in the intersection of all automorphism groups of the graphs $(\Gamma_i)_i$. The ***automorphism group*** of the association scheme $(\Omega, \mathscr{R})$ is defined to be the group

$$\mathrm{Aut}(\mathscr{R}) := \bigcap_{i=1}^{d} \mathrm{Aut}(\Gamma_i). \tag{1}$$

Equivalently, $\mathrm{Aut}(\mathscr{R})$ can also be viewed as the matrix group

$$\{P \in \mathrm{M}_\Omega(\mathbb{C}) : P \text{ is a permutation matrix such that } A_iP = PA_i, \text{ for } 1 \leqslant i \leqslant d\}.$$

The Terwilliger algebra of an association scheme is an important tool introduced by Paul Terwilliger in [19, 20, 21] to study the so-called P- and Q-polynomial association schemes. As such, these algebras were first introduced for these types of schemes, but nevertheless, they can also be defined broadly on any association schemes. The drawback is however that the structure of these algebras become much more complex in the non-P- and non-Q-polynomial cases. For examples, the Terwilliger algebras of the group association scheme of $\mathrm{Sym}(n)$ is only well-understood for $3 \leqslant n \leqslant 7$; see [1, 3, 10]. Thus, other structures that approximate the Terwilliger algebra are very important.

We formally define what a Terwilliger algebra now. Given a d-class association scheme $(\Omega, \mathscr{R} = \{R_i\}_{i=0}^d)$ and a vertex $\omega \in \Omega$, for any $0 \leqslant i \leqslant d$, we define the diagonal matrix $E^*_{i,\omega}$ such that

$$E^*_{i,\omega}(\alpha, \alpha) = \begin{cases} 1 & \text{if } (\omega, \alpha) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

The ***Terwilliger algebra*** of the association scheme $(\Omega, \mathscr{R})$ with respect to a vertex $\omega \in \Omega$ is the subalgebra of $\mathrm{M}_\Omega(\mathbb{C})$ given by

$$T_\omega = \langle A_0, A_1, \ldots, A_d, E^*_{0,\omega}, E^*_{1,\omega}, \ldots, E^*_{d,\omega} \rangle. \tag{2}$$

Note that two Terwilliger algebras defined with respect to two vertices need not be isomorphic. However, if ω and ω' are in the same orbit of $\mathrm{Aut}(\mathscr{R})$, then T_ω and $T_{\omega'}$ are isomorphic $\mathbb{C}$-algebras. For any $g \in \mathrm{Aut}(\mathscr{R})$, we let P_g be the corresponding permutation matrix in $\mathrm{M}_\Omega(\mathbb{C})$. Two important structures related to the Terwilliger algebra T_ω are its subspace $T_{0,\omega}$ and the centralizer algebra $\tilde{T}_\omega$. The former is defined as the subspace

$$T_{0,\omega} = \mathrm{Span}_{\mathbb{C}} \left\{ E^*_{i,\omega} A_j E^*_{k,\omega} : 0 \leqslant i, j, k \leqslant d \right\},$$

while the latter is

$$\tilde{T}_\omega = \{A \in \mathrm{M}_\Omega(\mathbb{C}) : P_gA = AP_g, g \in \mathrm{Aut}(\mathscr{R})_\omega\}.$$

In other words, $\tilde{T}_\omega$ is the centralizer algebra of the group stabilizer $\mathrm{Aut}(\mathscr{R})_\omega$ of ω in $\mathrm{Aut}(\mathscr{R})$. It is immediate that $T_{0,\omega} \subseteq T_\omega$. The adjacency matrices of the graphs $(\Gamma_i)_i$ are clearly in $\tilde{T}_\omega$ by definition of $\mathrm{Aut}(\mathscr{R})$, and it is not hard to show that $E^*_i \in \tilde{T}_\omega$, for any $0 \leqslant i \leqslant d$; see [10]. Therefore, for any $\omega \in \Omega$, we have

$$T_{0,\omega} \subseteq T_\omega \subseteq \tilde{T}_\omega. \tag{3}$$

If $T_{0,\omega} = T_\omega = \tilde{T}_\omega$, then the Terwilliger algebra enjoys both combinatorial and algebraic structures from $T_{0,\omega}$ and $\tilde{T}_\omega$, respectively. Indeed, the equality $T_{0,\omega} = T_\omega$ implies that the subspace $T_{0,\omega}$ is closed under multiplication, which turns out to be a property equivalent to the association scheme $(\Omega, \mathscr{R})$ being triply regular (see [3, 17]). The algebra $\tilde{T}_\omega$ admits a

basis consisting of matrices corresponding to orbits of $\mathrm{Aut}(\mathscr{R})_\omega$ acting on $\Omega\times\Omega$. Moreover, it is semisimple and its Wedderburn decomposition can be obtained directly from the action of the point stabilizer $\mathrm{Aut}(\mathscr{R})_\omega$. Therefore, the equality $T_\omega = \tilde{T}_\omega$ imposes very strong algebraic structures on the Terwilliger algebra.

Whenever $\mathrm{Aut}(\mathscr{R})$ is transitive on Ω and $T_{0,\omega} = T_\omega = \tilde{T}_\omega$, then we say that the association scheme $(\Omega, \mathscr{R})$ is ***triply transitive***. This notion was first introduced by Bannai and Munemasa in [3], where they studied the Terwilliger algebras of association schemes arising from conjugacy classes of groups. The automorphism groups of such association schemes are always transitive, so the definition given in [3] only requires $T_{0,\omega} = T_\omega = \tilde{T}_\omega$. Other papers on triply transitive association schemes arising from groups have also appeared in [1, 4, 5, 9, 10, 14, 23, 24]. Recently, works on this property for more general association schemes have also appeared in the literature [11, 13, 15].

Given a finite group G with identity 1, we may define an association scheme from the conjugacy classes of G. If $C_0 = \{1\}, C_1, \ldots, C_d$ are the distinct conjugacy classes of G, then define

$$R_i = \{(\alpha, \beta) \in G \times G : \alpha^{-1}\beta \in C_i\} \tag{4}$$

for $0 \leqslant i \leqslant d$. Then, it is easy to see that $(G, \mathscr{R})$, with $\mathscr{R} = \{R_i\}_{i=0}^d$, is a commutative association scheme. This association scheme is the ***conjugacy class scheme*** or the ***group association scheme*** of G. Group schemes are important classes of association schemes since much information on them can be extracted using the underlying algebraic structure—the group. For instance, their character tables or first eigenmatrices can be computed directly using the normalized irreducible characters of G. They also have applications in the study of Erdős-Ko-Rado type problems (see [8]).

The group G acts regularly as a subgroup of $\mathrm{Aut}(\mathscr{R})$ by left multiplication on itself. Hence, $\mathrm{Aut}(\mathscr{R})$ is always transitive, and thus the Terwilliger algebras T_g and $T_{g'}$ are isomorphic for any $g, g' \in G$. Consequently, we will always define the Terwilliger algebras of group association schemes with respect to the identity element 1 of G. Moreover, we will henceforth use the notations $T_0 := T_{0,1}$, $T := T_1$, and $\tilde{T} := \tilde{T}_1$.

This paper focuses on properties of the group association schemes arising from the Möbius group[1] $\mathrm{PGL}_2(q)$; see Section 3 for the definition. For any prime power q, let $\mathscr{R}_q$ be the relations defined in (4) with $G = \mathrm{PGL}_2(q)$. First, we determine the automorphism group of the scheme $(\mathrm{PGL}_2(q), \mathscr{R}_q)$. The first main result is stated below.

Theorem 1.1. *For any prime power q, we have*

$$\mathrm{Aut}(\mathscr{R}_q) = \left(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\right) \rtimes \langle\psi\rangle,$$

where $L_{\mathrm{PGL}_2(q)}$ and $R_{\mathrm{PGL}_2(q)}$ are respectively the left-regular and right-regular representations of $\mathrm{PGL}_2(q)$, and ψ is the anti-automorphism mapping $x \in \mathrm{PGL}_2(q)$ to its inverse.

In order to prove Theorem 1.1, we first find the automorphism group of a graph defined in relation to the Erdős-Ko-Rado theorem of $\mathrm{PGL}_2(q)$, and then we deduce $\mathrm{Aut}(\mathscr{R}_q)$ from this group. The graph in question is the derangement graph of the group $\mathrm{PGL}_2(q)$ in its natural transitive action on $\mathrm{PG}_1(q)$. We note that derangement graphs can be defined for any transitive permutation group (see [8]), but for our purpose, we only consider the one for $\mathrm{PGL}_2(q)$ in its action on $\mathrm{PG}_1(q)$. Recall that a derangement in a finite transitive group $G \leqslant$

[1] also known as the projective general linear group of degree 2

$\mathrm{Sym}(\Omega)$ is a fixed-point-free permutation. The ***derangement graph*** $\Gamma_{\mathrm{PGL}_2(q)}$ of $\mathrm{PGL}_2(q) \leqslant \mathrm{Sym}(\mathrm{PG}_1(q))$ is the Cayley graph $\mathrm{Cay}(\mathrm{PGL}_2(q), D_{\mathrm{PGL}_2(q)})$, where

$$D_{\mathrm{PGL}_2(q)} = \{g \in \mathrm{PGL}_2(q) : g(\omega) \neq \omega, \text{ for all } \omega \in \mathrm{PG}_1(q)\}.$$

That is, $\Gamma_{\mathrm{PGL}_2(q)}$ is the Cayley graph of $\mathrm{PGL}_2(q)$ whose connection set consists of all derangements of $\mathrm{PGL}_2(q)$ in its natural action on $\mathrm{PG}_1(q)$. It is immediate that the graph $\Gamma_{\mathrm{PGL}_2(q)}$ is the union of graphs from $(\mathrm{PGL}_2(q), \mathscr{R}_q)$, corresponding to conjugacy classes of derangements. The automorphism group of the derangement graph $\Gamma_{\mathrm{PGL}_2(q)}$ is stated in the next theorem.

Theorem 1.2. *For any power q of a prime number, we have*

$$\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) = \left(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\right) \rtimes \left(\langle\psi\rangle \times \gamma_{\mathrm{Aut}(\mathbb{F}_q)}\right),$$

where:

- $L_{\mathrm{PGL}_2(q)}$ *is the left-regular representation of* $\mathrm{PGL}_2(q)$,
- $R_{\mathrm{PGL}_2(q)}$ *is the right-regular representation of* $\mathrm{PGL}_2(q)$,
- $\gamma_{\mathrm{Aut}(\mathbb{F}_q)}$ *is the group of conjugation by elements of the field automorphism* $\mathrm{Aut}(\mathbb{F}_q)$, *and*
- $\psi : \mathrm{PGL}_2(q) \to \mathrm{PGL}_2(q)$ *is the anti-automorphism mapping an element to its inverse.*

Next, we focus on the triple transitivity of the group association scheme of $\mathrm{PGL}_2(q)$. We use Theorem 1.1 to determine certain properties of the centralizer algebra $\tilde{T}$, and this enables us to classify all q for which the scheme $(\mathrm{PGL}_2(q), \mathscr{R}_q)$ is triply transitive.

Theorem 1.3. *The association scheme $(\mathrm{PGL}_2(q), \mathscr{R}_q)$ is triply transitive if and only if $q = 2$.*

This paper is organized as follows. In Section 2, we give some background results on group actions and automorphism group of schemes. In Section 3, we recall some important properties of $\mathrm{PGL}_2(q)$ that will be used in the proof of the main results. Section 4 is devoted to finding the automorphism group of the derangement graph of $\mathrm{PGL}_2(q)$, and Section 5 deals with the proof of Theorem 1.1. Finally, we give a proof of Theorem 1.3 in Section 6.

2. Background results

2.1. **Group actions.** All groups considered in this paper are finite, and all digraphs are simple. All groups actions are left actions, that its, a group G with identity element 1 acts on a set Ω if there exists a map $\alpha : G \times \Omega \to \Omega$ such that for any $g, g' \in G$, we have $\alpha(gg', \omega) = \alpha(g, \alpha(g', \omega))$ and $\alpha(1, \omega) = \omega$, for $\omega \in \Omega$. The group G acts ***faithfully*** on Ω if $\{g \in G : \alpha(g, \omega) = \omega, \forall \omega \in \Omega\} = \{1\}$.

Given a group G, we define the maps $\lambda_g : G \to G$ and $\rho_g : G \to G$ such that $\lambda_g(x) = gx$ and $\rho_g(x) = xg^{-1}$, for $x \in G$. The sets $L_G = \{\lambda_g : g \in G\}$ and $R_G = \{\rho_g : g \in G\}$ are subgroups of $\mathrm{Sym}(G)$, and they are respectively called the ***left-regular representation*** and the ***right-regular representation*** of the group G. If K is an overgroup containing G as a normal subgroup, then for any $k \in K$, let $\gamma_k : G \to G$ such that $\gamma_k(x) = kxk^{-1}$ for $x \in G$. It is not hard to see that $\gamma_k = \lambda_k\rho_k$ for any $k \in K$. The group L_G and R_G commute, and if $Z(G)$ is trivial, then $\langle L_G, R_G\rangle = L_G \times R_G$. See [10] for details.

2.2. **Automorphism groups of group association schemes.** Let G be a group with identity element denoted by 1. Denote the conjugacy classes of G by $C_0 = \{1\}, C_1, \ldots, C_d$. If $\mathscr{R} = \{R_0, R_1, \ldots, R_d\}$, where $R_i = \{(g, h) : g^{-1}h \in C_i\}$ is the set defined in (4), then we have seen before that $(G, \mathscr{R})$ is a commutative association scheme — the group association scheme of G. The digraph $\Gamma_i = (G, R_i)$ is clearly the Cayley digraph $\mathrm{Cay}(G, C_i)$ for $1 \leqslant i \leqslant d$.

Therefore, $L_G \leqslant \mathrm{Aut}(\Gamma_i)$ for all $1 \leqslant i \leqslant d$. Since C_i is a conjugacy class, the connection set of Γ_i is closed under conjugation, and consequently, $R_G \leqslant \mathrm{Aut}(\Gamma_i)$ for all $1 \leqslant i \leqslant d$.

If $\psi : G \to G$ such that $\psi(x) = x^{-1}$ for $x \in G$, then $\psi(C_i)$ is also a conjugacy class for $1 \leqslant i \leqslant d$. Thus, for any $1 \leqslant i \leqslant d$, it is clear that ψ is an automorphism of Γ_i if and only if C_i is closed under taking inverse. We conclude that if G admits a non-inverse closed conjugacy class, then $\psi \notin \mathrm{Aut}(\mathscr{R})$. For any $g \in G$, we note that $\psi\lambda_g\psi^{-1} = \rho_{g^{-1}}$ and $\psi\rho_g\psi^{-1} = \lambda_{g^{-1}}$. In other words, ψ normalizes $\langle L_G, R_G\rangle$.

We deduce that the group $\langle L_G, R_G, \psi\rangle \leqslant \mathrm{Aut}(\Gamma_i)$ for all $1 \leqslant i \leqslant d$, whenever the conjugacy classes of G are inverse closed. Hence, we obtain the following lemma.

Lemma 2.1. *Let G be a group whose centre is trivial, and whose conjugacy classes are inverse closed. If $(G, \mathscr{R})$ is its group association scheme, then*

$$(L_G \times R_G) \rtimes \langle\psi\rangle \leqslant \mathrm{Aut}(\mathscr{R}).$$

3. The projective general linear groups of degree two

Let $q = p^k$ be the power of a prime p, where $k \geqslant 1$ is an integer. The finite field on q elements is denoted by $\mathbb{F}_q$. We denote the projective line over $\mathbb{F}_q$, i.e. the set of all 1-dimensional subspaces of $\mathbb{F}_q^2$, by $\mathrm{PG}_1(q)$. The elements of $\mathrm{PG}_1(q)$ are the $q+1$ subspaces given by

$$\left\langle \begin{bmatrix} x \\ 1 \end{bmatrix} \right\rangle, \ x \in \mathbb{F}_q \text{ and } \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle.$$

Thus, we may view $\mathrm{PG}_1(q)$ as $\mathbb{F}_q$ with an additional point at infinity, i.e., $\mathrm{PG}_1(q) = \mathbb{F}_q \cup \{\infty\}$.

The projective general linear group $\mathrm{PGL}_2(q)$ is the group of all ***Möbius transformations***, that is, transformations of the form $h : \mathrm{PG}_1(q) \to \mathrm{PG}_1(q)$ such that

$$h(z) = \frac{az+b}{cz+d}$$

where $ad - cd \neq 0$, and with the rules: $\frac{1}{\infty} = 0$, $\frac{c}{0} = \infty$ for $c \neq 0$ and $h(\infty) = \frac{a}{c}$.

3.1. **Cross-ratios and orbits.** The group $\mathrm{PGL}_2(q)$ acts transitively on $\mathrm{PG}_1(q)$. The stabilizer of $\infty \in \mathrm{PG}_1(q)$ in this action of $\mathrm{PGL}_2(q)$ is a Borel subgroup of $\mathrm{PGL}_2(q)$, which is therefore of order $q(q-1)$, isomorphic to $\mathrm{AGL}_1(q)$. This subgroup acts transitively on the remaining q points of $\mathrm{PG}_1(q)$, so the action of $\mathrm{PGL}_2(q)$ on $\mathrm{PG}_1(q)$ is also 2-transitive[2]. In addition, it is well known that this action of $\mathrm{PGL}_2(q)$ is sharply 3-transitive, so for every triples (x, y, z) and (x', y', z') of elements of $\mathrm{PG}_1(q)$ with distinct entries, there exists a unique $g \in \mathrm{PGL}_2(q)$ such that $(g(x), g(y), g(z)) = (x', y', z')$.

Given four distinct elements $x, y, z, t \in \mathrm{PG}_1(q)$, their ***cross-ratio*** is

$$(x, y; z, t) := \frac{(z-x)(t-y)}{(z-y)(t-x)}.$$

The cross-ratio is an invariant of $\mathrm{PGL}_2(q)$ in the sense that if $g \in \mathrm{PGL}_2(q)$, then we have $(g(x), g(y); g(z), g(t)) = (x, y; z, t)$ for any distinct $x, y, z, t \in \mathrm{PG}_1(q)$. In fact, this invariant determines whether an element of $\mathrm{Sym}(\mathrm{PG}_1(q))$ is in $\mathrm{PGL}_2(q)$ or not.

[2]that is, it is also transitive on the set of ordered pairs of elements of $\mathrm{PG}_1(q)$ with distinct entries

Theorem 3.1. *A permutation of* $\mathrm{PG}_1(q)$ *belongs to* $\mathrm{PGL}_2(q)$ *if and only if it preserves the cross-ratio of any four distinct elements of* $\mathrm{PG}_1(q)$.

By sharp 3-transitivity, the group $\mathrm{PGL}_2(q)$ acts intransitively on the set all quadruples of $\mathrm{PG}_1(q)$ with distinct entries, whenever $q \geqslant 4$. Given two quadruples with distinct entries (u, v, w, t) and (u', v', w', t') of $\mathrm{PG}_1(q)$, there exists at most one element of $\mathrm{PGL}_2(q)$ mapping one to the other. If $g \in \mathrm{PGL}_2(q)$ is such that $(g(u), g(v), g(w), g(t)) = (u', v', w', t')$, then by Theorem 3.1 we have the cross-ratio equality

$$(u, v; w, t) = (u', v'; w', t').$$

Denote the Frobenius automorphism of $\mathbb{F}_q$ by θ, that is, $\theta : \mathbb{F}_q \to \mathbb{F}_q$ such that $\theta(x) = x^p$ for all $x \in \mathbb{F}_q$. It is well known that $\mathrm{Aut}(\mathbb{F}_q) = \mathrm{Aut}(\mathbb{F}_{p^k}/\mathbb{F}_p) = \langle\theta\rangle \cong \mathbb{Z}_k$, where $k = \log_p(q)$. The group $\mathrm{Aut}(\mathbb{F}_q)$ acts as automorphism of $\mathrm{PGL}_2(q)$ via the following action. For any $h : z \to \frac{az+b}{cz+d}$ in $\mathrm{PGL}_2(q)$, we have

$$\theta \cdot h : z \mapsto \frac{\theta(a)z + \theta(b)}{\theta(c)z + \theta(d)}.$$

Then, the projective semilinear group of degree 2 over $\mathbb{F}_q$ is the group denoted by $\mathrm{P\Gamma L}_2(q) = \mathrm{PGL}_2(q) \rtimes \mathrm{Aut}(\mathbb{F}_q) = \mathrm{PGL}_2(q) \rtimes \langle\theta\rangle$ with multiplication

$$(g\theta^i)(h\theta^j) = g(\theta^i \cdot h)\theta^{i+j}$$

for any $g, h \in \mathrm{PGL}_2(q)$ and $0 \leqslant i, j \leqslant k-1$. We note that by the property of the semi-direct product, we have $\theta \cdot h = \theta h \theta^{-1}$, for any $h \in \mathrm{PGL}_2(q)$.

Next, we state an important theorem that will be used in the proof of the main result. This result can be derived from the Fundamental Theorem of Projective Geometry [12, Chapter 2(ii)] or from a group theoretic result from [7, Page 245].

Theorem 3.2. *We have* $\mathrm{N}_{\mathrm{Sym}(\mathrm{PG}_1(q))}(\mathrm{PGL}_2(q)) = \mathrm{P\Gamma L}_2(q)$.

3.2. **Conjugacy classes.** The conjugacy classes of $\mathrm{PGL}_2(q)$ are determined by the canonical forms of matrices in the group $\mathrm{GL}_2(q)$ of 2×2 invertible matrices with entries in $\mathbb{F}_q$. Note that $\mathrm{PGL}_2(q) \cong \mathrm{GL}_2(q)/Z(\mathrm{GL}_2(q))$, and the element $h \in \mathrm{PGL}_2(q)$ such that $h : z \mapsto \frac{az+b}{cz+d}$ corresponds to the congruence class of all scalar multiples of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Any element of $\mathrm{PGL}_2(q)$ is therefore conjugate to an element of the form:

(1) $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, where $\alpha \in \mathbb{F}_q^*$. The corresponding map is $z \mapsto \frac{z}{\alpha}$.

(2) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, corresponding to the map $z \mapsto z + 1$.

(3) $\begin{pmatrix} 0 & -r^{q+1} \\ 1 & r + r^q \end{pmatrix}$, where $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. The corresponding map is $z \mapsto \frac{-r^{q+1}}{z+(r+r^q)}$.

We note that every conjugacy class of $\mathrm{PGL}_2(q)$ is closed under inversion. In particular, any $g \in \mathrm{PGL}_2(q)$ is conjugate to its inverse by an involution.

3.3. **Centralizer and normalizer of non-diagonalizable elements.** Let $h \in \mathrm{PGL}_2(q)$ such that $h : z \to z + 1$. Every element of order p is conjugate to h. The centralizer of h in $\mathrm{PGL}_2(q)$ is the subgroup

$$\mathrm{C}_{\mathrm{PGL}_2(q)}(\langle h\rangle) = E_q := \{g \in \mathrm{PGL}_2(q) \mid g : z \mapsto z + a : a \in \mathbb{F}_q\}.$$

Note that the group $\mathrm{C}_{\mathrm{PGL}_2(q)}(\langle h\rangle)$ is a Sylow p-subgroup of $\mathrm{PGL}_2(q)$. The size of the conjugacy class of $\mathrm{PGL}_2(q)$ containing h is therefore $\frac{q(q^2-1)}{q} = q^2-1$. Using matrix representation, it is not hard to see that the normalizer of $\langle h\rangle$ is the group

$$\mathrm{N}_{\mathrm{PGL}_2(q)}(\langle h\rangle) = \left\{g \in \mathrm{PGL}_2(q) \mid g : z \mapsto \tfrac{z+a}{b}, a \in \mathbb{F}_q, b \in \mathbb{F}_p^*\right\},$$

whenever q is odd. If q is even, then the normalizer of $\langle h\rangle$ is given by

$$\mathrm{N}_{\mathrm{PGL}_2(q)}(\langle h\rangle) = \{g \in \mathrm{PGL}_2(q) \mid g : z \mapsto z + a,\ a \in \mathbb{F}_q\} = E_q.$$

4. Proof of Theorem 1.2

Cocliques[3] in derangement graphs have important applications in Erdős-Ko-Rado combinatorics of transitive groups. In [16], it was shown that the cocliques of the derangement graph $\Gamma_{\mathrm{PGL}_2(q)}$, for the permutation group corresponding to the action of $\mathrm{PGL}_2(q)$ on $\mathrm{PG}_1(q)$, have size at most q^2-q. Moreover, a coclique of maximum size in $\Gamma_{\mathrm{PGL}_2(q)}$ must be of the form

$$S_{u,v} := \{g \in \mathrm{PGL}_2(q) : g(u) = v\},$$

for some $u, v \in \mathrm{PG}_1(q)$.

To prove Theorem 1.2, we first show that $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ contains a large subgroup isomorphic to

$$\left(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\right) \rtimes \langle \psi, \gamma_\theta\rangle$$

in § 4.1. The rest of the proof builds up step by step that this subgroup is in fact equal to $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$. In § 4.2, we show that $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ embed into $\mathrm{Sym}(\mathrm{PG}_1(q) \times \mathrm{PG}_1(q))$, and use this fact along with the intersection sizes of maximum cocliques to embed $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ into the automorphism group of the $(q+1) \times (q+1)$ grid graph or the Hamming graph $H(2, q+1)$ in § 4.3. Using all these results, we finally give a proof to Theorem 1.2 in § 4.4. We note that some of the arguments used in the early steps in §4.2 are inspired by arguments used in [6], however, the rest of the proof of Theorem 1.2 requires techniques that are unique to Möbius groups.

4.1. A large subgroup of automorphism.

We will show that $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ admits a large subgroup, which turns out to be the full automorphism group.

Since $\Gamma_{\mathrm{PGL}_2(q)}$ is a (left) Cayley graph, the celebrated result of Sabidussi [18] asserts that $L_{\mathrm{PGL}_2(q)} \leqslant \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$. The right-regular representation of need not be contained in the automorphism group of a Cayley graph, in general. However, in the case of $\Gamma_{\mathrm{PGL}_2(q)}$, the connection set is a union of conjugacy classes, and this implies $R_{\mathrm{PGL}_2(q)} \leqslant \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$. As $\mathrm{PGL}_2(q)$ is centreless, we have $\langle L_{\mathrm{PGL}_2(q)}, R_{\mathrm{PGL}_2(q)}\rangle = L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}$. See [10, § 3.1] for details. The same argument can be used to show that $\langle L_{\mathrm{P\Gamma L}_2(q)}, R_{\mathrm{P\Gamma L}_2(q)}\rangle = L_{\mathrm{P\Gamma L}_2(q)} \times R_{\mathrm{P\Gamma L}_2(q)}$, but as we will see later, not every element of this group is an automorphism of $\Gamma_{\mathrm{PGL}_2(q)}$.

Lemma 4.1. *We have* $\mathrm{Aut}(\mathbb{F}_q) \cong \langle \gamma_\theta\rangle \leqslant \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$.

[3] that is, set of vertices that are pairwise non-adjacent.

Proof. It is clear that $\mathrm{Aut}(\mathbb{F}_q) = \langle\theta\rangle$ is isomorphic to $\langle\gamma_\theta\rangle$. We note that $g \in \mathrm{PGL}_2(q)$ fixes $u \in \mathrm{PG}_1(q)$ if and only if $\theta \cdot g$ fixes $\theta(u)$. Hence, if $g \in \mathrm{PGL}_2(q)$ is a derangement on $\mathrm{PG}_1(q)$, then so is $\theta \cdot g$. For any $x, y \in \mathrm{PGL}_2(q)$, we have

$$\begin{aligned}\gamma_\theta(x) \sim \gamma_\theta(y) &\Leftrightarrow \gamma_\theta(x)^{-1}\gamma_\theta(y) \text{ has no fixed points in } \mathrm{PG}_1(q)\\ &\Leftrightarrow \gamma_\theta(x^{-1}y) = \theta(x^{-1}y)\theta^{-1} = \theta\cdot(x^{-1}y) \text{ has no fixed points in } \mathrm{PG}_1(q)\\ &\Leftrightarrow x^{-1}y \text{ has no fixed points in } \mathrm{PG}_1(q)\\ &\Leftrightarrow x \sim y.\end{aligned}$$

Hence, $\gamma_\theta \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$. □

Next, we define an action of $\mathrm{Aut}(\mathbb{F}_q)$ as automorphism on $L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}$. For any $g, g' \in \mathrm{PGL}_2(q)$, we have

$$\gamma_\theta(\lambda_g\rho_{g'})\gamma_\theta^{-1} = (\gamma_\theta\lambda_g\gamma_\theta^{-1})(\gamma_\theta\rho_{g'}\gamma_\theta^{-1}) = \lambda_{\theta\cdot g}\rho_{\theta\cdot g'}.$$

Therefore, $L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)} \trianglelefteq \langle L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}, \gamma_\theta\rangle$. If $\phi \in \langle\gamma_\theta\rangle \cap (L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)})$ is non-trivial, then $\phi = \gamma_\theta^i = \lambda_g\rho_{g'}$ for some $0 \leqslant i \leqslant k-1$ and $g, g' \in \mathrm{PGL}_2(q)$. Evaluating $\gamma_\theta^i = \lambda_g\rho_{g'}$ on the identity of $\mathrm{PGL}_2(q)$, we have $g' = g$, and consequently, $\lambda_g\rho_{g'} = \lambda_g\rho_g = \gamma_g$. We deduce that $\gamma_\theta^i = \gamma_{\theta^i} = \gamma_g$. As $\mathrm{PGL}_2(q) \cap \langle\gamma_\theta\rangle$ is trivial, we deduce that $i = 0$ and $g = 1$, and thus ϕ is the identity element. Consequently, we obtain the next lemma.

Lemma 4.2. *We have* $\langle L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}, \gamma_\theta\rangle = \left(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\right) \rtimes \langle\gamma_\theta\rangle$.

Recall that $\psi : \mathrm{PGL}_2(q) \to \mathrm{PGL}_2(q)$ is the map such that $\psi(x) = x^{-1}$ for any $x \in \mathrm{PGL}_2(q)$. Since the centre of $\mathrm{PGL}_2(q)$ is trivial, it follows that $\langle L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}, \psi\rangle = \left(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\right) \rtimes \langle\psi\rangle$. Moreover, for $x \in \mathrm{PGL}_2(q)$, we have

$$\theta\psi(x) = \theta(x^{-1}) = \left(x^{-1}\right)^p = (x^p)^{-1} = \psi(x^p) = \psi\theta(x).$$

In other words, ψ and θ commute. We deduce the following lemma whose proof is omitted.

Lemma 4.3. *For any q, we have*

$$\left(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\right) \rtimes (\langle\psi\rangle \times \langle\gamma_\theta\rangle) \leqslant \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}).$$

For the remainder of the section, we show that equality in fact holds in Lemma 4.3.

4.2. **An induced action on** $\mathrm{PG}_1(q) \times \mathrm{PG}_1(q)$**.** For any $u, v \in \mathrm{PG}_1(q)$, recall that

$$S_{u,v} = \{g \in \mathrm{PGL}_2(q) : g(u) = v\}. \tag{5}$$

We note that $S_{u,u}$ is the stabilizer of $u \in \mathrm{PG}_1(q)$ in $\mathrm{PGL}_2(q)$, and for any $v \in \mathrm{PG}_1(q)$, the set $S_{u,v}$ is a coset of $S_{u,u}$.

For any triples (u_1, u_2, u_3) and (v_1, v_2, v_3) of $\mathrm{PG}_1(q)$ with distinct entries, we have

$$|S_{u_1,v_1} \cap S_{u_2,v_2} \cap S_{u_3,v_3}| = 1$$

due to the fact that $\mathrm{PGL}_2(q)$ is sharply 3-transitive. As discussed before, $\mathrm{PGL}_2(q)$ acts intransitively on the set of quadruple of elements of $\mathrm{PG}_1(q)$ with distinct entries, for $q \geqslant 4$, and two quadruples (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) of elements of $\mathrm{PG}_1(q)$ with distinct

entries belong to the same orbit if and only if they have the same cross-ratio. Therefore, we deduce that

$$|S_{u_1,v_1} \cap S_{u_2,v_2} \cap S_{u_3,v_3} \cap S_{u_4,v_4}| = 1 \text{ if and only if } (u_1, u_2; u_3, u_4) = (v_1, v_2; v_3, v_4).$$

Let $\Omega = \{S_{u,v} : u, v \in \mathrm{PG}_1(q)\}$. As Ω is the set of all cocliques of maximum size in $\Gamma_{\mathrm{PGL}_2(q)}$, the automorphism group $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ acts on Ω.

Lemma 4.4. *The action of* $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ *on* Ω *is faithful.*

Proof. Assume that $\phi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ such that $\phi(S_{u,v}) = S_{u,v}$ for all $u, v \in \mathrm{PG}_1(q)$. Then, given $g \in \mathrm{PGL}_2(q)$ we have

$$\phi(\{g\}) = \phi\left(\bigcap_{u \in \mathrm{PG}_1(q)} S_{u,g(u)}\right) \subseteq \bigcap_{u \in \mathrm{PG}_1(q)} \phi(S_{u,g(u)}) = \bigcap_{u \in \mathrm{PG}_1(q)} S_{u,g(u)} = \{g\}.$$

Hence, $\phi(g) = g$ for all $g \in \mathrm{PGL}_2(q)$. Since $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ acts faithfully on the vertices of $\Gamma_{\mathrm{PGL}_2(q)}$, we must have that ϕ is the identity map. □

Thus, we have the embedding $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) \hookrightarrow \mathrm{Sym}(\Omega) \cong \mathrm{Sym}(\mathrm{PG}_1(q) \times \mathrm{PG}_1(q))$. Next, we examine the intersection of elements of Ω. If $\phi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$, then

$$|\phi(S_{u,v}) \cap \phi(S_{u',v'})| = |S_{u,v} \cap S_{u',v'}| \tag{6}$$

for any $u, v, u', v' \in \mathrm{PG}_1(q)$. Note that the latter holds more generally for permutations of $\mathrm{PGL}_2(q)$ that also permute Ω.

As previously discussed, the left-regular representation $L_{\mathrm{PGL}_2(q)} = \{\lambda_g : g \in \mathrm{PGL}_2(q)\}$ of $\mathrm{PGL}_2(q)$ is a subgroup of $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$. Its action on Ω is as follows. For $g \in \mathrm{PGL}_2(q)$ and $u, v \in \mathrm{PG}_1(q)$, we have $\lambda_g(S_{u,v}) = gS_{u,v} = S_{u,g(v)}$. Similarly, for $g \in \mathrm{PGL}_2(q)$, we have $\rho_g(S_{u,v}) = S_{u,v}g^{-1} = S_{g(u),v}$.

We determine the intersection sizes of elements of Ω in the next lemma.

Lemma 4.5. *If* $u, u', v, v' \in \mathrm{PG}_1(q)$, *then*

$$|S_{u,v} \cap S_{u',v'}| = \begin{cases} q(q-1) & \text{if } u = u' \text{ and } v = v' \\ 0 & \text{if } u = u' \text{ and } v \neq v', \text{ or } u \neq u' \text{ and } v = v' \\ q-1 & \text{if } (u,v) \neq (u',v'). \end{cases}$$

Proof. Clearly, if $u = u'$ and $v = v'$, then $S_{u,v} = S_{u',v'}$, so the size of their intersection is $q(q-1)$. The intersections $S_{u,v} \cap S_{u,v'}$ where $v \neq v'$ and $S_{u,v} \cap S_{u',v}$ where $u \neq u'$ are both empty, otherwise the elements would not be a map in the former, and would not be injective in the latter. Assume next that $(u, v) \neq (u', v')$. Let $g \in \mathrm{PGL}_2(q)$ such that $g(v) = u$. Then, by (6) we have

$$|S_{u,v} \cap S_{u',v'}| = |\lambda_g(S_{u,v}) \cap \lambda_g(S_{u',v'})| = \left|S_{u,g(v)} \cap S_{u',g(v')}\right| = \left|S_{u,u} \cap S_{u',g(v')}\right|. \tag{7}$$

Since $\mathrm{PGL}_2(q)$ is 2-transitive on $\mathrm{PG}_1(q)$, the stabilizer $S_{u,u}$ of $u \in \mathrm{PG}_1(q)$ has two orbits, namely $\{u\}$ and $\mathrm{PG}_1(q) \setminus \{u\}$. Note that since $(u, v) \neq (u', v')$, we know that $u', g(v') \in \mathrm{PG}_1(q) \setminus \{u\}$. Hence, there exists $h \in S_{u,u}$ such that $h(g(v')) = u'$. Using this in (7), we have

$$|S_{u,v} \cap S_{u',v'}| = |S_{u,u} \cap S_{u',g(v')}| = |hS_{u,u} \cap hS_{u',g(v')}| = |S_{u,u} \cap S_{u',u'}|.$$

Note that $S_{u,u} \cap S_{u',u'}$ is the pointwise stabilizer of u and u'. As $S_{u,u}$ acts transitively on $\mathrm{PG}_1(q) \setminus \{u\}$, by the orbit stabilizer lemma we have

$$|S_{u,u} \cap S_{u',u'}| = \frac{|S_{u,u}|}{|\mathrm{PG}_1(q) \setminus \{u\}|} = \frac{q(q-1)}{q} = q - 1.$$

Hence, we deduce that $|S_{u,v} \cap S_{u',v'}| = q - 1$. □

4.3. **Reduction to the** $(q+1)\times(q+1)$**–grid.** For any integers $d, q \geq 1$, the Hamming graph $H(d, n)$ is the graph whose vertex set consists of the d-tuples with entries from $\{0, 1, \ldots, q-1\}$, and where two d-tuples are adjacent if they differ in exactly one entry or position. The graph $H(2, n)$ is also known as the $n \times n$ grid graph.

From Lemma 4.5, we know that the intersection sizes take exactly three values. From this, we may define a graph that represents the structure of the intersection of elements of Ω. Let $\mathcal{G}_q$ be the graph whose vertex set is $\mathrm{PG}_1(q) \times \mathrm{PG}_1(q)$, where two vertices (u, v) and (u', v') are adjacent if $|S_{u,v} \cap S_{u',v'}| = 0$. We illustrate $\mathcal{G}_3$ below.

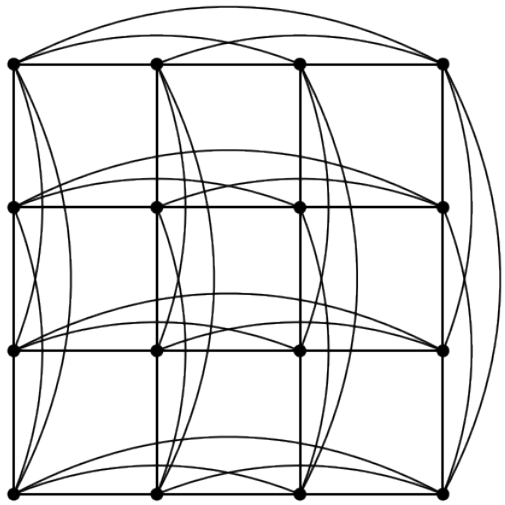


FIGURE 1. The graph $\mathcal{G}_3$.

As the group $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ acts (faithfully) on Ω, it also acts (faithfully) on the vertices $\mathcal{G}_q$. To obtain this action, we view any $\phi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) \hookrightarrow \mathrm{Sym}(\Omega)$ as a mapping on $\mathrm{PG}_1(q) \times \mathrm{PG}_1(q)$ to itself. We write

$$\phi(u, v) = (u', v') \Leftrightarrow \phi(S_{u,v}) = S_{u',v'} \tag{8}$$

for any $u, v, v', v' \in \mathrm{PG}_1(q)$ and $\phi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$. In fact, $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ does not only act faithfully on $\mathrm{PG}_1(q) \times \mathrm{PG}_1(q)$, it also acts as automorphism of $\mathcal{G}_q$.

Lemma 4.6. *Any* $\phi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ *induces an automorphism of* $\mathcal{G}_q$. *In particular,* $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ *embeds into* $\mathrm{Aut}(\mathcal{G}_q)$.

Proof. By Lemma 4.4, ϕ induces a permutation on $\mathrm{PG}_1(q) \times \mathrm{PG}_1(q)$. Given two vertices (u, v) and (u', v') of $\mathcal{G}_q$, we have

$$\begin{aligned}(u,v) \sim_{\mathcal{G}_q} (u',v') &\Leftrightarrow S_{u,v} \cap S_{u',v'} = \varnothing \\ &\Leftrightarrow \phi(S_{u,v}) \cap \phi(S_{u',v'}) = \varnothing \qquad (\text{ by (6)}) \\ &\Leftrightarrow \phi(u,v) \sim_{\mathcal{G}_q} \phi(u',v').\end{aligned}$$

In other words, ϕ induces an automorphism of $\mathcal{G}_q$. The rest of the proof is immediate. □

$\phi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$	$\overline{\phi} \in \mathrm{Aut}(\mathcal{G}_q)$	Comments
λ_g	$(1, g; 1)$	$g \in \mathrm{PGL}_2(q)$
ρ_g	$(g, 1; 1)$	$g \in \mathrm{PGL}_2(q)$
ψ	$(1, 1; -1)$	swaps coordinates
γ_θ	$(\theta, \theta; 1)$	$\theta \in \mathrm{Aut}(\mathbb{F}_q)$

TABLE 1. Images of certain automorphisms of $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) \hookrightarrow \mathrm{Aut}(\mathcal{G}_q)$.

Using Lemma 4.5, the graph $\mathcal{G}_q$ is clearly isomorphic the Hamming graph $H(2, q+1) = K_{q+1} \square K_{q+1}$. Consequently, we have $\mathrm{Aut}(\mathcal{G}_q) = \mathrm{Sym}(\mathrm{PG}_1(q)) \wr C_2 = (\mathrm{Sym}(\mathrm{PG}_1(q)) \times \mathrm{Sym}(\mathrm{PG}_1(q))) \rtimes C_2$, which consists of all triples

$$(\sigma, \tau; r) \in \mathrm{Sym}(\mathrm{PG}_1(q)) \times \mathrm{Sym}(\mathrm{PG}_1(q)) \times C_2$$

such that

$$(\sigma, \tau; r)(i, j) = \begin{cases} (\sigma(i), \tau(j)) & \text{if } r = 1 \\ (\tau(j), \sigma(i)) & \text{if } r = -1. \end{cases}$$

As $\mathrm{Sym}(\mathrm{PG}_1(q))$ is centreless, we have

$$\langle L_{\mathrm{Sym}(\mathrm{PG}_1(q))}, R_{\mathrm{Sym}(\mathrm{PG}_1(q))}, \psi \rangle = \left(L_{\mathrm{Sym}(\mathrm{PG}_1(q))} \times R_{\mathrm{Sym}(\mathrm{PG}_1(q))}\right) \rtimes \langle \psi \rangle.$$

The group $\left(L_{\mathrm{Sym}(\mathrm{PG}_1(q))} \times R_{\mathrm{Sym}(\mathrm{PG}_1(q))}\right) \rtimes \langle \psi \rangle$ acts on Ω by permuting the indices in Ω. Define the map

$$\begin{aligned}\Phi : \left(L_{\mathrm{Sym}(\mathrm{PG}_1(q))} \times R_{\mathrm{Sym}(\mathrm{PG}_1(q))}\right) \rtimes \langle \psi \rangle &\longrightarrow (\mathrm{Sym}(\mathrm{PG}_1(q)) \times \mathrm{Sym}(\mathrm{PG}_1(q))) \rtimes C_2 \\ \lambda_\sigma \rho_\tau \psi^i &\longmapsto (\tau, \sigma; (-1)^i).\end{aligned}$$

The map Φ is clearly a group isomorphism. Let $\Phi(\phi) = \overline{\phi}$ is the permutation induced by $\phi \in (L_{\mathrm{Sym}(\mathrm{PG}_1(q))} \times R_{\mathrm{Sym}(\mathrm{PG}_1(q))}) \rtimes \langle \psi \rangle$ in $\mathrm{PG}_1(q) \times \mathrm{PG}_1(q)$. For any subgroup $M \leqslant (L_{\mathrm{Sym}(\mathrm{PG}_1(q))} \times R_{\mathrm{Sym}(\mathrm{PG}_1(q))}) \rtimes \langle \psi \rangle$, we define $\overline{M} = \Phi(M)$. We give the image under Φ of certain elements of $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ in its embedding into $\mathrm{Aut}(\mathcal{G}_q)$ in Table 1; the proof is straightforward, so we omit it.

We note that any $(\sigma, \tau; (-1)^i) \in \mathrm{Sym}(\mathrm{PG}_1(q)) \wr C_2$ can be expressed as the product

$$(\sigma, \tau; (-1)^i) = (\sigma, \tau; 1)(1, 1; (-1)^i) = (\sigma, \tau; 1)\overline{\psi}^i.$$

Since $\psi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$, it is an easy exercise to show that there exists a subgroup $K \leqslant \mathrm{Sym}(\mathrm{PG}_1(q)) \times \mathrm{Sym}(\mathrm{PG}_1(q)) \leq \mathrm{Sym}(\mathrm{PG}_1(q)) \wr C_2$ such that

$$\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) \cong \overline{\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})} = K\langle \overline{\psi} \rangle = K \rtimes \langle \overline{\psi} \rangle. \tag{9}$$

In particular,

$$\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) = \Phi^{-1}(K) \rtimes \langle \psi \rangle. \tag{10}$$

4.4. **Proof of Theorem 1.2.** In this section, we combine the results from the previous sections to determine the subgroup $\Phi^{-1}(K)$, or equivalently, $K \leqslant \mathrm{Sym}(\mathrm{PG}_1(q)) \times \mathrm{Sym}(\mathrm{PG}_1(q))$.

For any $g \in \mathrm{PGL}_2(q)$, recall from Table 1 that $\overline{\lambda_g} = (1, g; 1)$ and $\overline{\rho_g} = (g, 1; 1)$. Moreover, we have $\overline{\gamma_\theta} = (\theta, \theta; 1)$. It is therefore clear from Lemma 4.2 that

$$(\overline{L}_{\mathrm{PGL}_2(q)} \times \overline{R}_{\mathrm{PGL}_2(q)}) \rtimes \langle \overline{\gamma_\theta} \rangle \leqslant K, \tag{11}$$

where K is the subgroup defined in (9). It is worth noting that a typical element of $(\overline{L}_{\mathrm{PGL}_2(q)} \times \overline{R}_{\mathrm{PGL}_2(q)}) \rtimes \langle \overline{\gamma_\theta} \rangle$ is of the form $(g\theta^i, g'\theta^i; 0)$, where $g, g' \in \mathrm{PGL}_2(q)$ and $0 \leqslant i, i' \leqslant k-1$.

Now, let $\phi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ such that $\overline{\phi} \in K$. As $K \leq \mathrm{Sym}(\mathrm{PG}_1(q)) \times \mathrm{Sym}(\mathrm{PG}_1(q)) \leqslant \mathrm{Sym}(\mathrm{PG}_1(q)) \wr C_2$, there exist two permutations $\sigma, \tau \in \mathrm{Sym}(\mathrm{PG}_1(q))$ such that

$$\overline{\phi} = (\sigma, \tau; 1)\,.$$

Claim 1. The permutations σ and τ lie in $\mathrm{P\Gamma L}_2(q)$.

Proof Claim 1. For any $\varphi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ and for any pair of quadruples (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) with distinct entries from $\mathrm{PG}_1(q)$, we have

$$|S_{u_1,v_1} \cap S_{u_2,v_2} \cap S_{u_3,v_3} \cap S_{u_4,v_4}| = |\varphi(S_{u_1,v_1}) \cap \varphi(S_{u_2,v_2}) \cap \varphi(S_{u_3,v_3}) \cap \varphi(S_{u_4,v_4})|.$$

Let us show that $\sigma g \sigma^{-1}, \tau g \tau^{-1} \in \mathrm{PGL}_2(q)$, for any $g \in \mathrm{PGL}_2(q)$. For any four distinct elements u_1, u_2, u_3 and u_4 of $\mathrm{PG}_1(q)$, using the fact that ϕ and ρ_g are automorphisms of $\Gamma_{\mathrm{PGL}_2(q)}$, we have

$$\begin{aligned}
1 &= \left|\bigcap_{i=1}^4 S_{u_i,u_i}\right| = \left|\bigcap_{i=1}^4 \phi^{-1}(S_{u_i,u_i})\right| = \left|\bigcap_{i=1}^4 (S_{\overline{\phi^{-1}}(u_i,u_i)})\right| = \left|\bigcap_{i=1}^4 S_{\sigma^{-1}(u_i),\tau^{-1}(u_i)}\right| \\
&= \left|\bigcap_{i=1}^4 \rho_g(S_{\sigma^{-1}(u_i),\tau^{-1}(u_i)})\right| = \left|\bigcap_{i=1}^4 S_{g\sigma^{-1}(u_i),\tau^{-1}(u_i)}\right| = \left|\bigcap_{i=1}^4 \phi(S_{g\sigma^{-1}(u_i),\tau^{-1}(u_i)})\right| \\
&= \left|\bigcap_{i=1}^4 (S_{\overline{\phi}(g\sigma^{-1}(u_i),\tau^{-1}(u_i))})\right| = \left|\bigcap_{i=1}^4 S_{\sigma g\sigma^{-1}(u_i),\tau\tau^{-1}(u_i)}\right| = \left|\bigcap_{i=1}^4 S_{\sigma g\sigma^{-1}(u_i),u_i}\right|.
\end{aligned}$$

Hence, there is a unique element of $\mathrm{PGL}_2(q)$ that maps the quadruples (u_1, u_2, u_3, u_4) to $(\sigma g\sigma^{-1}(u_1), \sigma g\sigma^{-1}(u_2), \sigma g\sigma^{-1}(u_3), \sigma g\sigma^{-1}(u_4))$. By Theorem 3.1, we must have

$$(u_1, u_2; u_3, u_4) = \left(\sigma g\sigma^{-1}(u_1), \sigma g\sigma^{-1}(u_2); \sigma g\sigma^{-1}(u_3), \sigma g\sigma^{-1}(u_4)\right),$$

for any quadruples (u_1, u_2, u_3, u_4) of $\mathrm{PG}_1(q)$ with distinct entries. In other words, $\sigma g\sigma^{-1} \in \mathrm{PGL}_2(q)$ for any $g \in \mathrm{PGL}_2(q)$. Thus, $\sigma \in \mathrm{N}_{\mathrm{Sym}(\mathrm{PG}_1(q))}(\mathrm{PGL}_2(q))$. Similarly, we can show that $\tau \in \mathrm{N}_{\mathrm{Sym}(\mathrm{PG}_1(q))}(\mathrm{PGL}_2(q))$ by using the automorphism $\lambda_g \in L_{\mathrm{PGL}_2(q)}$. The result now follows from Theorem 3.2. □

From Claim 1, we deduce that $K \leqslant \overline{L}_{\mathrm{P\Gamma L}_2(q)} \times \overline{R}_{\mathrm{P\Gamma L}_2(q)}$. Therefore, for $\varphi \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$, we have

$$\overline{\varphi} = (g, g'; (-1)^i) = \overline{\lambda_{g'}}\,\overline{\rho_g}\,\overline{\psi^i}$$

for some $g, g' \in \mathrm{P\Gamma L}_2(q)$ and $i \in \{0,1\}$. Since Φ is an isomorphism, we note that $\Phi^{-1}(K) \leq L_{\mathrm{P\Gamma L}_2(q)} \times R_{\mathrm{P\Gamma L}_2(q)}$, and by Lemma 4.3, we have

$$\left(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\right) \rtimes \langle \gamma_\theta \rangle \leqslant \Phi^{-1}(K) \leqslant \left(L_{\mathrm{P\Gamma L}_2(q)} \times R_{\mathrm{P\Gamma L}_2(q)}\right). \tag{12}$$

Next, we determine the elements of $L_{\mathrm{P\Gamma L}_2(q)} \times R_{\mathrm{P\Gamma L}_2(q)}$ that lie in $\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$.

Claim 2. Let $g, g' \in \mathrm{PGL}_2(q)$ and $0 \leqslant i, j \leqslant k-1$. Then, $\rho_{g'\theta^j}\lambda_{g\theta^i} \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$ if and only if $i = j$. In this case, $\rho_{g'\theta^j}\lambda_{g\theta^i} = \rho_{g'}\lambda_g\gamma_{\theta^i} \in (L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}) \rtimes \langle \gamma_\theta \rangle$.

Proof of Claim 2. The converse of the statement follows from Lemma 4.3. Now, assume that $\rho_{g'\theta^j}\lambda_{g\theta^i} \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$. It is immediate that

$$\rho_{g'\theta^j}\lambda_{g\theta^i} \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) \text{ if and only if } \rho_{\theta^j}\lambda_{\theta^i} \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}).$$

As $\rho_{\theta^j}\lambda_{\theta^i} \in \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)})$, it is a permutation of $\mathrm{PGL}_2(q)$. In particular, $\rho_{\theta^j}\lambda_{\theta^i}(x) \in \mathrm{PGL}_2(q)$ for any $x \in \mathrm{PGL}_2(q)$. Since $\rho_{\theta^j}\lambda_{\theta^i}(x) = \theta^i x \theta^{-j}$, for $x \in \mathrm{PGL}_2(q)$, we have $\theta^i \mathrm{PGL}_2(q)\theta^{-j} \subseteq \mathrm{PGL}_2(q)$. The latter can only happen if $i = j$. □

We deduce from Claim 2 that the elements of $\Phi^{-1}(K)$ must be of the form

$$\lambda_{g\theta^i}\rho_{g'\theta^i} = \lambda_g\rho_{g'}\lambda_{\theta^i}\rho_{\theta^i} = \lambda_g\rho_{g'}\gamma_{\theta^i} \in (L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}) \rtimes \langle \gamma_\theta \rangle$$

for some $g, g' \in \mathrm{PGL}_2(q)$ and $0 \leqslant i \leqslant k-1$. By (12), we have

$$\Phi^{-1}(K) = (L_{\mathrm{PGL}_2(q) \times R_{\mathrm{PGL}_2(q)}}) \rtimes \langle \gamma_\theta \rangle.$$

Therefore, we conclude that

$$\mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) = \Phi^{-1}(K) \rtimes \langle \psi \rangle = (L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}) \rtimes \langle \gamma_\theta, \psi \rangle,$$

thus proving Theorem 1.2.

5. Proof of Theorem 1.1

In order to prove Theorem 1.1, we use Theorem 1.2. Let $C_0 = \{1\}, C_1, \ldots, C_d$ be the conjugacy classes of $\mathrm{PGL}_2(q)$, and let $I = \{i : C_i \text{ consists of derangements}\}$. Recall that $(\mathrm{PGL}_2(q), \mathscr{R}_q)$ is the conjugacy class association scheme, where $\mathscr{R}_q = \{R_0, R_1, \ldots, R_d\}$ whose elements are defined in (4). By definition, we have

$$\mathrm{Aut}(\mathscr{R}_q) = \bigcap_{i=1}^{d} \mathrm{Aut}(\Gamma_i).$$

We note that the derangement graph $\Gamma_{\mathrm{PGL}_2(q)}$ is the union of all graphs in $\{\Gamma_i : i \in I\}$. On the one hand, we have

$$\begin{aligned}
\mathrm{Aut}(\mathscr{R}_q) &= \bigcap_{i=1}^{d} \mathrm{Aut}(\Gamma_i) \\
&\leqslant \bigcap_{i \in I} \mathrm{Aut}(\Gamma_i) \\
&\leqslant \mathrm{Aut}(\Gamma_{\mathrm{PGL}_2(q)}) \\
&= \left(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\right) \rtimes \langle \psi, \gamma_\theta \rangle \quad \text{by Theorem 1.2.}
\end{aligned}$$

Recall that the conjugacy classes of $\mathrm{PGL}_2(q)$ are closed under taking inverse. On the other hand, by Lemma 2.1, we know that

$$\big(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\big) \rtimes \langle \psi \rangle \leqslant \mathrm{Aut}(\mathscr{R}).$$

Combining the two expressions, we have

$$(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}) \rtimes \langle \psi \rangle \leqslant \mathrm{Aut}(\mathscr{R}) \leqslant (L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}) \rtimes \langle \psi, \gamma_\theta \rangle. \tag{13}$$

By (13), an element $\lambda_g \rho_{g'} \psi^i \gamma_{\theta^j} \in (L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}) \rtimes \langle \psi, \gamma_\theta \rangle$ belongs to $\mathrm{Aut}(\mathscr{R})$ if and only if $\gamma_{\theta^j} \in \mathrm{Aut}(\mathscr{R})$. We claim that $\gamma_{\theta^j} \in \mathrm{Aut}(\mathscr{R})$ only if $j = 0$. Assume by contradiction that $\gamma_{\theta^j} \in \mathrm{Aut}(\mathscr{R})$ where $1 \leqslant j \leqslant k-1$. By definition, we have $\gamma_{\theta^j} \in \mathrm{Aut}(\Gamma_i)$ for all $1 \leqslant i \leqslant d$. In addition, since γ_{θ^j} fixes the identity of $\mathrm{PGL}_2(q)$, it is clear that γ_{θ^j} must leave all conjugacy classes of $\mathrm{PGL}_2(q)$ invariant. Since $1 \leqslant j \leqslant k-1$, there exists $d \in \mathbb{F}_q$ such that is $\theta^j(d) = d^{p^j} \neq d$. Considering the element $g \in \mathrm{PGL}_2(q)$ such that $g : x \mapsto \frac{1}{dx}$, we have $\theta^j \cdot g : x \mapsto \frac{1}{d^{p^j} x}$. It is immediate that $\theta^j \cdot g$ and g cannot be conjugate since $1 \leqslant j \leqslant k-1$, and we conclude that θ^j does not preserve the conjugacy classes of $\mathrm{PGL}_2(q)$. Hence, we obtain a contradiction, and we must have $j = 0$. Using (13), we have $\mathrm{Aut}(\mathscr{R}) = \big(L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}\big) \rtimes \langle \psi \rangle$. This completes the proof.

6. Proof of Theorem 1.3

Recall that the conjugacy classes of $\mathrm{PGL}_2(q)$ are given in §3.2. Denote the conjugacy classes of $\mathrm{PGL}_2(q)$ by $C_0, C_1, \ldots, C_d$ where $g_0 = 1 \in C_0, g_1 \in C_1, \ldots, g_d \in C_d$ are distinct representatives. As before, let R_i be the relation corresponding to the conjugacy class C_i as defined in (4) for $0 \leqslant i \leqslant d$. Finally, let $\mathscr{R}_q = \{R_0, R_1, \ldots, R_d\}$.

Suppose that $1 \leqslant \mathbf{i} \leqslant d$ is such that $C_\mathbf{i}$ is the conjugacy class of containing $h : z \mapsto z+1$. By Theorem 1.1, we know that $\mathrm{Aut}(\mathscr{R}) = (L_{\mathrm{PGL}_2(q)} \times R_{\mathrm{PGL}_2(q)}) \rtimes \langle \psi \rangle$. The stabilizer of the identity element of $\mathrm{PGL}_2(q)$ in $\mathrm{Aut}(\mathscr{R})$ is the subgroup

$$K = \big\{\lambda_g \rho_g \psi^j : g \in \mathrm{PGL}_2(q) \text{ and } 0 \leqslant j \leqslant 1\big\} = \mathrm{Inn}(\mathrm{PGL}_2(q)) \times \langle \psi \rangle.$$

Therefore, K acts transitively on every conjugacy class of $\mathrm{PGL}_2(q)$.

As $g_\mathbf{i}$ and h are in the same conjugacy class, we know that $\dim(E_\mathbf{i}^* \tilde{T} E_\mathbf{i}^*)$ is the number of orbits of K contained in $C_{g_\mathbf{i}} \times C_{g_\mathbf{i}}$; see [15]. By transitivity of K on $C_\mathbf{i}$, this number is exactly the number of orbits of the subgroup of K fixing h on the set $C_{g_\mathbf{i}}$. Let us now find the subgroup K_h.

If q is odd, then let $\mathbf{g} : z \mapsto a^{-1}z$, where $a \in \mathbb{F}_p^*$ is a primitive element, and let $h^* = \mathbf{g}^{\frac{q-1}{2}}$. We note that $h^* h (h^*)^{-1} = h^{-1}$, and $\mathrm{N}_{\mathrm{PGL}_2(q)}(\langle h \rangle) = E_q \rtimes \langle \mathbf{g} \rangle$; see §3.3. Since $(\gamma_{h^*}\psi)\gamma_g(\psi^{-1}\gamma_{h^*}^{-1}) = \gamma_{h^* g (h^*)^{-1}}$ for any $g \in E_q$, we deduce that $\langle \gamma_{h^*}\psi \rangle$ normalizes the subgroup $\{\gamma_g : g \in E_q\}$, and it is not hard to see that

$$\{\gamma_g : g \in E_q\}\langle \gamma_{h^*}\psi \rangle = \{\gamma_g : g \in E_q\} \rtimes \langle \gamma_{h^*}\psi \rangle.$$

If q is even, then h is an involution, and by §3.3, we know that $\mathrm{N}_{\mathrm{PGL}_2(q)}(\langle h \rangle) = E_q$. For any $g \in \mathrm{PGL}_2(q)$, we have $\psi \gamma_g \psi^{-1} = \gamma_g$, and we deduce that

$$\{\gamma_g : g \in E_q\}\langle \psi \rangle = \{\gamma_g : g \in E_q\} \times \langle \psi \rangle.$$

In the next lemma, we show that the above subgroups are the stabilizers of h in K.

Lemma 6.1. *The following statements hold.*
(a) If q is odd, then $K_h = \{\gamma_g : g \in E_q\} \rtimes \langle \gamma_{h^*}\psi \rangle$.

(b) If q is even, then $K_h = \{\gamma_g : g \in E_q\} \times \langle\psi\rangle$.

Proof. (a) Assume that q is odd. It is immediate that $\gamma_g \in K_h$ for any $g \in E_q$, and $\gamma_{h^*}\psi(h) = \gamma_{h^*}(h^{-1}) = h$, so $\gamma_{h^*}\psi \in K_h$. Consequently, we have $\{\gamma_g : g \in E_q\} \rtimes \langle\gamma_{h^*}\psi\rangle \leqslant K_h$.

Assume that $\gamma_g\psi^i \in K_h$ for some $g \in \mathrm{PGL}_2(q)$ and $0 \leqslant i \leqslant 1$. If $i = 0$, then $h = \gamma_g(h) = ghg^{-1}$, thus $g \in E_q$. In this case, $\gamma_g\psi^i \in \{\gamma_g : g \in E_q\}$. If $i = 1$, then $h = \gamma_g\psi(h) = gh^{-1}g^{-1}$, and thus $g \in E_qh^*$. In this case, there exists $x \in E_q$ such that $g = xh^*$, and $\gamma_g\psi^i = \gamma_{xh^*}\psi = \gamma_x\gamma_{h^*}\psi \in \{\gamma_g : g \in E_q\} \rtimes \langle\gamma_{h^*}\psi\rangle$. We deduce that $K_h = \{\gamma_g : g \in E_q\} \rtimes \langle\gamma_{h^*}\psi\rangle$.

(b) Assume that q is even. It is immediate that $\{\gamma_g : g \in E_q\} \times \langle\psi\rangle \leqslant K_h$. Given $\gamma_g\psi^i \in K_h$ with $g \in E_q$ and $0 \leqslant i \leqslant 1$, we have $\gamma_g\psi^i(h) = \gamma_g(h) = ghg^{-1} = h$. Therefore, we have $K = \{\gamma_g : g \in E_q\} \times \langle\psi\rangle$. □

Now, we give a lower bound on the number of orbits of K_h on $C_{\mathbf{i}}$.

Lemma 6.2. *The group K_h has at least $\frac{3}{2}(q-1)$ orbits on $C_{\mathbf{i}}$.*

Proof. Since every element of order p is conjugate to h, every non-identity element of E_q is conjugate to h. Now, it is immediate that K_h fixes every non-identity element of E_q, which determines $q-1$ orbits. Noting that $|C_{\mathbf{i}}| = q^2 - 1$, the number of orbits of K_h on $C_{\mathbf{i}}$ is therefore at least

$$(|E_q| - 1) - \frac{|C_{\mathbf{i}}|}{|K_h|} = \begin{cases} q - 1 + \frac{q^2-1-(q-1)}{2q} = \frac{3}{2}(q-1) & \text{if } q \text{ is odd,} \\ q - 1 + \frac{q^2-1-(q-1)}{2q} = \frac{3}{2}(q-1) & \text{if } q \text{ is even.} \end{cases}$$

This completes the proof. □

Proof of Theorem 1.3. Assume that the association scheme $(\mathrm{PGL}_2(q), \mathscr{R}_q)$ is triply transitive, that is, $T_0 = \tilde{T}$.

Suppose that q is odd. As $E_i^*A_jE_i^* \neq 0 \Leftrightarrow p_{ij}^i \neq 0$, the dimension of $E_i^*T_0E_i^*$ is therefore equal the size of the set $\{j : p_{ij}^i \neq 0\}$. Therefore, $\dim E_{g_i}^*T_0E_{g_i}^*$ is at most the number of conjugacy classes of $\mathrm{PGL}_2(q)$, which is equal to $q+2$. By Lemma 6.2, we deduce

$$\dim E_i^*T_0E_i^* \leqslant q + 2 < \frac{3}{2}(q-1) \leqslant \dim E_i^*\tilde{T}E_i^*,$$

whenever $q > 7$. The case $q = 7$ can be verified using `Sagemath` [22]. For $q = 7$, we have $\dim T_0 = 278$ whereas $\dim \tilde{T} = 391$. In particular, we have $\dim E_i^*T_0E_i^* = 5$ and $\dim E_i^*\tilde{T}E_i^* = 12$. For $q = 5$, since $\mathrm{PGL}_2(5) \cong \mathrm{Sym}(5)$, by [1, Theorem 1], we deduce that $T_0 \neq \tilde{T}$. For $q = 3$, we have $\mathrm{PGL}_2(3) \cong \mathrm{Sym}(4)$, and by [3], we have $T_0 \neq \tilde{T}$.

Suppose that q is now even. Using Lemma 6.2, we have

$$\dim E_i^*T_0E_i^* \leqslant q + 1 < \frac{3}{2}(q-1) \leqslant \dim E_i^*\tilde{T}E_i^*,$$

whenever $q > 5$. For $q = 4$, we have $\mathrm{PGL}_2(4) \cong \mathrm{Alt}(5)$ and $\dim E_i^*\tilde{T}E_i^* = 6$, whereas $\dim E_i^*T_0E_i^* = 5$. For $q = 2$, we have $\mathrm{PGL}_2(2) \cong \mathrm{Sym}(3)$. It was shown in [3] that the association scheme of the latter is triply transitive.

Consequently, if $(\mathrm{PGL}_2(q), \mathscr{R})$ is triply transitive, then $q = 2$. The converse holds immediately. □

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