DISTANCE-REGULAR GRAPHS ARISING FROM THE TRANSITIVE ACTION OF $\mathrm{PSL}_2(q)$ ON 2-SUBSETS OF $\mathrm{PG}_1(q)$

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Abstract. Let Γ be a distance-regular graph whose automorphim group $\mathrm{Aut}(\Gamma)$ has rank at most 11 and admits a subgroup which is permutation equivalent to $\mathrm{PSL}_2(q)$ acting on 2-subsets of $\mathrm{PG}_1(q)$. It is shown that either $\mathrm{Aut}(\Gamma)$ does not preserve the edges of the Kneser graph $K(q+1,2)$ or the graph Γ is one of: the Kneser graph $K(q+1,2)$, the Johnson graph $J(q+1,2)$, the line graph of the Petersen graph, the Coxeter graph or the line graph of the Tutte-Coxeter graph.

1. Introduction

Distance-regular graphs are highly regular graphs that play a central role in algebraic graph theory. Given a connected graph $\Gamma = (\Omega, E)$ and a vertex $v \in \Omega$, the eccentricity $\varepsilon = \varepsilon(v)$ of v is the maximum distance between v and any other vertex of Γ. The diameter of Γ is the largest eccentricity of a vertex of Γ. The distance partition of Γ with respect to v is the partition $\pi(v) = \{\Gamma_0(v), \Gamma_1(v), \ldots, \Gamma_\varepsilon(v)\}$, where

$$\Gamma_i(v) := \{u \in \Omega : d(v,u) = i\},$$

for $0 \leq i \leq \varepsilon$. We say that Γ is a ***distance-regular graph*** of diameter D if there exist some non-negative integers $(a_i)_{0\leq i\leq D}, (b_i)_{0\leq i\leq D-1}, (c_i)_{1\leq i\leq D}$ such that for any $v \in V$ and $u \in \Gamma_i(v)$, we have

$$\begin{aligned} a_i &= |\{w \in \Gamma_i(v) : \{u,w\} \in E\}|, \\ b_i &= |\{w \in \Gamma_{i+1}(v) : \{u,w\} \in E\}|, \\ c_i &= |\{w \in \Gamma_{i-1}(v) : \{u,w\} \in E\}|. \end{aligned}$$

The numbers $(a_i), (b_i), (c_i)$ are called the intersection numbers of Γ. Note that the distance-regular graph Γ is regular with valency $k = b_0 = |\Gamma_1(v)|$, and $a_i + b_i + c_i = k$, for $0 \leq i \leq D$. A distance-regular graph with diameter 2 is called a ***strongly-regular***

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graph. In particular, a strongly-regular graph with parameters (n, k, λ, μ) is a graph on n vertices, where every vertex has degree k, every pair of adjacent vertices has λ common neighbours, and every pair of non-adjacent vertices has μ common neighbours.

Distance-regular graphs have been extensively studied in the literature, and their study is a very active research area; see [3, 4, 11, 16] for more information on these graphs. Distance-regular graphs can arise from many discrete structures such as a geometry, permutation groups, codes, and designs. Hence, classifying distance-regular graphs with given parameters or constructing new ones is an interesting problem since these can lead to a better understanding of the underlying discrete structures.

One of many ways that distance-regular graphs can arise is via transitive permutation groups. Since such distance-regular graphs are always vertex transitive, let us first recall some definitions and facts about vertex-transitive graphs. Given a transitive permutation group $G \leq \mathrm{Sym}(\Omega)$, the ***orbitals*** of G are the orbits of G in its induced action on $\Omega \times \Omega$. The ***rank*** of $G \leq \mathrm{Sym}(\Omega)$ is the number of orbitals of G. For an orbital $(\omega, \omega')^G$ not equal to the diagonal orbital $(\omega, \omega)^G$, the corresponding ***orbital digraph*** is the digraph with vertex set Ω, and arc set equal to $(\omega, \omega')^G$. An orbital $(\omega, \omega')^G$ of G is called ***self-paired*** if $(\omega, \omega')^G = (\omega', \omega)^G$. If $(\omega, \omega')^G$ is self-paired, then the corresponding orbital digraph may be viewed as a simple and undirected graph. We will hereafter refer to the orbital digraphs corresponding to self-paired orbitals as ***orbital graphs***. If $(\omega, \omega')^G \neq (\omega', \omega)^G$, then $(\omega, \omega')^G$ and $(\omega', \omega)^G$ are called paired orbitals. If Γ is a graph obtained by taking a union of orbital graphs and paired orbital digraphs of a transitive group $G \leq \mathrm{Sym}(\Omega)$, then Γ is called a G***-vertex-transitive*** graph. As $G \leq \mathrm{Aut}(\Gamma)$ is transitive in this case, the graph Γ is vertex transitive. Conversely, for any vertex-transitive graph Γ, it is well known that, for every transitive group $G \leq \mathrm{Aut}(\Gamma)$, the graph Γ is G-vertex-transitive (see [9, Proposition 1.4.6]).

Distance-regular graphs such as the Hamming graphs, Johnson graphs, and Grassmann graphs are examples arising from transitive group actions. In contrast, distance-regular graphs such as the Tutte 12-cage, the Gritsenko strongly-regular graph with parameters $(65, 32, 15, 16)$, and many block intersection graph of designs do not arise from transitive groups. In general, a distance-regular graph $\Gamma = (\Omega, E)$ with a transitive automorphism group is always G-vertex-transitive, for any transitive subgroup $G \leq \mathrm{Aut}(\Gamma)$. However, a distance-regular graph arising from a proper transitive subgroup of $\mathrm{Aut}(\Gamma)$ need not be equal to Γ. For instance, this situation occurs for the action of $\mathrm{PSL}_2(7)$ on the 2-subsets of $\mathrm{PG}_1(7)$, giving rise to the Kneser graph $K(8, 2)$, which is a strongly-regular graph, and the Coxeter graph, which is a diameter 4 distance-regular graph. Similarly, the Kneser graph $K(10, 2)$ and the line graph of the Tutte-Coxeter graph both arise from the action of $\mathrm{PSL}_2(9)$ on 2-subsets of $\mathrm{PG}_1(q)$.

This paper is concerned with distance-regular graphs that can arise from these actions of $\mathrm{PSL}_2(q)$ on the 2-subsets of $\mathrm{PG}_1(q)$ via the induced action on $\mathrm{PG}_1(q)$. Recall that a transitive group $G \leq \mathrm{Sym}(\Omega)$ is called 2***-homogeneous*** if it acts transitively on the 2-subsets of Ω via the induced action given by $\{\omega, \delta\}^g := \{\omega^g, \delta^g\}$, for any distinct $\delta, \omega \in \Omega$ and $g \in G$. Let $\Omega^{\{2\}}$ be the collection of all 2-subsets of Ω. As $\mathrm{Sym}(\Omega)$ is 2-homogeneous for $|\Omega| \geq 4$, let us consider the transitive action of $\mathrm{Sym}(\Omega)$ on the $\Omega^{\{2\}}$. This action of $\mathrm{Sym}(\Omega)$ on $\Omega^{\{2\}}$ is faithful of rank 3, and the corresponding orbitals are

$$
\begin{aligned}
O_0 &= \{(U, U) : U \in \Omega^{\{2\}}\}, \\
O_1 &= \{(U, V) : U, V \in \Omega^{\{2\}}, |U \cap V| = 1\}, \\
O_2 &= \{(U, V) : U, V \in \Omega^{\{2\}}, |U \cap V| = 0\}.
\end{aligned} \tag{1}
$$

We will refer to the orbitals $O_0, O_1,$ and O_2 in (1) as the ***canonical orbitals*** of the set Ω. The orbital graphs corresponding to O_1 and O_2 are respectively the Johnson graph $J(\Omega,2) \cong J(|\Omega|,2)$ and the Kneser graph $K(\Omega,2) \cong K(|\Omega|,2)$. The automorphism group of $K(\Omega,2)$, where $|\Omega| \geq 5$, is the group $\mathrm{Sym}(\Omega)$ acting on $\Omega^{\{2\}}$. Any 2-homogeneous group $G \leq \mathrm{Sym}(\Omega)$ naturally preserves the orbitals O_1 and O_2, and so must also be a subgroup of the automorphism $\mathrm{Sym}(\Omega)$ of $K(\Omega,2)$. In particular, the permutation representation of $\mathrm{PSL}_2(q)$ acting on 2-subsets of $\mathrm{PG}_1(q)$ is a subgroup of $\mathrm{Sym}(\mathrm{PG}_1(q))$ on 2-subsets of $\mathrm{PG}_1(q)$. Hence, for any prime power q, the Johnson graph $J(q+1,2)$ and the Kneser graph $K(q+1,2)$ are $\mathrm{PSL}_2(q)$-vertex-transitive, for the action of $\mathrm{PSL}_2(q)$ on 2-subsets of $\mathrm{PG}_1(q)$. As we have seen before, the two sporadic examples which are the Coxeter graph and the line graph of the Tutte-Coxeter graph also arise from this action of $\mathrm{PSL}_2(q)$ for $q = 7$ and $q = 9$, respectively.

At a BIRS workshop on Movement and Symmetry in Graphs in November 2024, Robert Bailey asked the following question.

Question 1.1. Except for the Coxeter graph and the line graph of the Tutte-Coxeter graph, are $J(q+1,2)$ and $K(q+1,2)$ the only distance-regular graphs arising from the action of $\mathrm{PSL}_2(q)$ on 2-subsets of $\mathrm{PG}_1(q)$?

The answer to this question turns out to be negative, and an infinite family of examples is given in [6]. In [6], De Caen and van Dam showed that the orbital graphs of $\mathrm{PGL}_2(q)$ in its action on 2-subsets of $\mathrm{PG}_1(q)$ consist of a graph isomorphic to $J(q+1,2)$ together with $\frac{q-3}{2}$ or $\frac{q-4}{2}$ spanning subgraphs of $K(q+1,2)$, depending on whether q is odd or q is even respectively. When $q = 2^{2k}$ for some small values of the integer $k \geq 2$, it was noted in [6] that there is a 4-class association scheme obtained by fusing these orbital graphs, in which $J(q+1,2)$ is isolated[1]. In addition, the union of $J(q+1,2)$ and two of these graphs yields a strongly-regular graph which is neither $J(q+1,2)$ nor $K(q+1,2)$. De Caen and van Dam conjectured that this fusion of the orbital graphs of $\mathrm{PGL}_2(q)$ always yields this 4-class association for every $q = 2^{2k}$, where $k \geq 2$. They noted that if the conjecture is true, then this strongly-regular graph always exists. This conjecture of De Caen and van Dam was subsequently proved in [10, 14]. Furthermore, it was shown in [10] that these strongly-regular graphs are isomorphic to the ones constructed by Brouwer and Wilbrink in [17, Section 7.B]. These are the strongly-regular graphs $NO_5^+(2^f)$ defined in [5, Section 3.1.4].

As $\mathrm{PGL}_2(q) = \mathrm{PSL}_2(q)$ whenever q is even, the strongly-regular graph $NO_5^+(2^f)$, for $f \geq 2$, gives a negative answer to Question 1.1. Note that since the above-mentioned strongly-regular graphs $NO_5^+(2^f)$ defined on $\Omega = \mathrm{PG}_1(4^f)$ were obtained by fusing $J(\Omega,2)$ with other orbital graphs contained in $K(\Omega,2)$, its automorphism group $O_5(2^f) \leq \mathrm{Aut}(NO_5^+(2^f))$ does not preserve the canonical orbitals of Ω (see (1)). In this paper, we determine all distance-regular graphs arising from the action of $\mathrm{PSL}_2(q)$ on 2-subsets of $\mathrm{PG}_1(q)$ with automorphism group of relatively small rank and preserving the canonical orbitals of $\mathrm{PG}_1(q)$. Our main result is stated as follows.

Theorem 1.2. *Let q be a prime power, $\Omega = \mathrm{PG}_1(q)$, and let Γ be a distance-regular graph arising from the action of $\mathrm{PSL}_2(q)$ on 2-subsets of Ω. If the automorphism group $\mathrm{Aut}(\Gamma)$ is of rank at most 11, then $\mathrm{Aut}(\Gamma)$ does not preserve the canonical orbitals of Ω, or one of the following cases occurs.*

(i) $q = 3$ and Γ is the complete multipartite graph $K_{2,2,2} \cong J(4,2)$,
(ii) $q = 5$ and Γ is the line graph of the Petersen graph, $J(6,2)$, or $K(6,2)$,
(iii) $q = 7$ and Γ is the Coxeter graph, $J(8,2)$, or $K(8,2)$,

[1] i.e., a class that is not merged with other relations in the association scheme

(iv) $q = 9$ *and* Γ *is the line graph of the Tutte-Coxeter graph,* $J(10, 2)$*, or* $K(10, 2)$*,*
(v) $q \geq 11$ *and* Γ *is* $J(q + 1, 2)$ *or* $K(q + 1, 2)$*.*

The 4-class association scheme which is a fusion of the orbital scheme of $\mathrm{PGL}_2(q)$ acting on 2-subsets of $\mathrm{PG}_1(q)$, when q is a power of 2, is the only known non-trivial fusion of the association scheme of this action of $\mathrm{PGL}_2(q)$. When q is odd, the group $\mathrm{PSL}_2(q)$ is a proper subgroup of $\mathrm{PGL}_2(q)$, so we ask the following question.

Question 1.3. Are the graphs $NO_5^+(2^f)$ the only distance-regular graphs arising from the action of $\mathrm{PSL}_2(q)$ on 2-subsets of $\mathrm{PG}_1(q)$, for $q \geq 13$, whose automorphism groups do not preserve the canonical orbitals of $\mathrm{PG}_1(q)$?

This paper is organized as follows. In Section 2, we give all the necessary background results needed in the proof. We recall some important properties of the group $\mathrm{PSL}_2(q)$ in Section 3. In Section 4, we prove Theorem 1.2.

2. Background results

2.1. **Association schemes.** Let Ω be a finite set and $R_0 = \{(\omega, \omega) : \omega \in \Omega\}, R_1, \ldots, R_d$ be a set of relations of Ω. The pair $(\Omega, \{R_i\}_{i=0,\ldots,d})$ is called an ***association scheme*** if

(i) $\{R_i\}_{i=0,\ldots,d}$ is a partition of $\Omega \times \Omega$,
(ii) there exists a permutation $* \in \mathrm{Sym}(d)$ of order dividing 2 such that for any $1 \leq i \leq d$, $R_{i^*} = \{(\omega, \delta) : (\delta, \omega) \in R_i\}$,
(iii) there exist some numbers $(p_{ij}^k)_{0 \leq i,j,k \leq d}$ such that for any $(\omega, \delta) \in R_k$,
$$|\{\alpha \in \Omega : (\omega, \alpha) \in R_i, (\alpha, \delta) \in R_j\}| = p_{ij}^k,$$
(iv) $p_{ij}^k = p_{ji}^k$ for all $0 \leq i, j, k \leq d$.

If $(\Omega, \{R_i\}_{i=0,\ldots,d})$ is an association scheme, then its order is the integer $|\Omega|$. The association scheme $(\Omega, \{R_i\}_{i=0,\ldots,d})$ is called a d-class association scheme. For any $1 \leq i \leq d$, the pair (Ω, R_i) is a digraph on Ω. If $1 \leq i \leq d$ such that $i^* = i$, then (Ω, R_i) can be viewed as an undirected graph. If $* \in \mathrm{Sym}(d)$ is trivial, then $(\Omega, \{R_i\}_{i=0,\ldots,d})$ is a ***symmetric*** association scheme. If only the conditions (i)-(iii) are satisfied, then we say that $(\Omega, \{R_i\}_{i=0,\ldots,d})$ is a ***homogeneous coherent configuration***. Note that a symmetric homogeneous coherent configuration must be an association scheme since (iv) is always satisfied.

Now, let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group, and let $O_0, O_1, \ldots, O_d$ be the orbitals of G. Then, the pair $(\Omega, \{O\}_{i=0,\ldots,d})$ satisfies (i)-(iii), so it is a homogeneous coherent configuration. Property (iv) is satisfied by $(\Omega, \{O_i\}_{i=0,\ldots,d})$ if and only if the permutation character of $G \leq \mathrm{Sym}(\Omega)$ is multiplicity free, that is, it is a sum of distinct irreducible characters of G (see [2]). As we have seen before, the rank of the permutation group G is equal to the number of orbitals, which is $d + 1$ in this case.

Assume now that $\Gamma = (\Omega, E)$ is a distance-regular graph of diameter D. Let $v \in \Omega$. Define the relations $R_i = \{u \in \Omega : d(v, u) = i\}$, for any $0 \leq i \leq D$. Then, $(\Omega, \{R_i\}_{i=0,\ldots,D})$ is a D-class symmetric association scheme. The association scheme $(\Omega, \{R_i\}_{i=0,\ldots,D})$ has an additional property called the P-polynomial property (see [3] for details). If in addition, $\Gamma = (\Omega, E)$ is a distance-regular graph with diameter D admitting a transitive automorphism group, then a vertex stabilizer in $\mathrm{Aut}(\Gamma)$ has at least $D + 1$ orbits, so the rank of G is at least $D + 1$. For example, the rank is exactly equal to the diameter plus one for distance-regular graphs whose automorphism groups are also distance transitive. However, the rank of the automorphism group is not necessarily equal to the diameter plus one, as in the case of the Shrikhande graph.

2.2. **Permutation groups.** Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group. A block of $G \leq \mathrm{Sym}(\Omega)$ is a subset $B \subset \Omega$ with the property that $B^g = B$ or $B^g \cap B = \varnothing$, for all $g \in G$. The subsets Ω and $\{\omega\}$, for any $\omega \in \Omega$, are always blocks of G, and they are referred as the trivial blocks of G. If every block of $G \leq \mathrm{Sym}(\Omega)$ is trivial, then G is called ***primitive***, otherwise, it is called ***imprimitive***. A useful characterization of primitive groups is that their point stabilizers are maximal subgroups (see [8, Corollary 1.5A]).

Recall that the rank of G is the number of orbitals of G. Equivalently, the rank is also the number of orbits of G_ω, for $\omega \in \Omega$. Define $r(G,\Omega)$ to be the rank of $G \leq \mathrm{Sym}(\Omega)$. Since $r(G,\Omega)$ is the number of orbits of G_ω, each orbit has size at most $|G_\omega|$. Moreover, $\{\omega\}$ is an orbit of G_ω. Therefore, we have $1+(r(G,\Omega)-1)|G_\omega| \geq |\Omega|$, and so we establish the relation

$$r(G,\Omega) \geq 1 + \frac{|\Omega|-1}{|G_\omega|}. \tag{2}$$

We say that $G \leq \mathrm{Sym}(\Omega)$ is 2-***transitive*** if G is transitive on the set $\Omega^{(2)} = \{(\omega,\omega') \in \Omega\times\Omega : \omega \neq \omega'\}$. In addition, $G \leq \mathrm{Sym}(\Omega)$ is called 2-***homogeneous*** if it acts transitively on the collection of all 2-subsets of Ω. It is clear that if G is 2-transitive, then it is 2-homogeneous, but the converse need not hold. We recall the following result about 2-homogeneous groups.

Theorem 2.1 ([8]). *Suppose that G is 2-homogeneous on Ω with $|\Omega| \geq 4$. Then, G is 2-transitive with the exception of*

$$\mathrm{ASL}_1(q) \trianglelefteq G \leq \mathrm{A\Gamma L}_1(q), \tag{3}$$

with $q \equiv 3 (\mathrm{mod}\, 4)$.

From the above theorem, we know that the vast majority of 2-homogeneous groups are 2-transitive. Recall that the socle of G is the subgroup $\mathrm{Soc}(G)$ generated by all minimal normal subgroups of G. In the next theorem, we recall the classification of 2-transitive groups, with respect to their socles.

Theorem 2.2. *A finite 2-transitive group is either of affine type*[2] *or almost-simple type*[3].

(i) If $\mathrm{Soc}(G) = \mathbb{F}_q^k$, then $G = \mathbb{F}_q^k \rtimes H$, where H is an irreducible subgroup of $\mathrm{GL}_k(q)$.
(ii) If G is of almost-simple type, then $\mathrm{Soc}(G)$ is one of the groups given in Table 1.

Recall that $\Omega^{\{2\}}$ is the collection of all 2-subsets of Ω. If $G \leq \mathrm{Sym}(\Omega)$ is 2-homogeneous, then G acts transitively on $\Omega^{\{2\}}$. Since this action is via the induced action of G on Ω, the group G also acts faithfully on $\Omega^{\{2\}}$. The orbitals of $\mathrm{Sym}(\Omega)$ on $\Omega^{\{2\}}$ are the canonical orbitals of Ω, which are the sets $O_0 = \{(U,U) : U \in \Omega^{\{2\}}\}$, $O_1 = \{(U,V) : U,V \in \Omega^{\{2\}}, |U\cap V| = 1\}$, $O_2 = \{(U,V) : U,V \in \Omega^{\{2\}}, |U \cap V| = 0\}$. These orbitals of $\mathrm{Sym}(\Omega)$ on $\Omega^{\{2\}}$ are preserved by any 2-homogeneous group $G \leq \mathrm{Sym}(\Omega)$. We will now prove the converse of this fact.

Lemma 2.3. *Let $G \leq \mathrm{Sym}(\Omega^{\{2\}})$ be a transitive group. Then, G preserves the canonical orbitals O_0, O_1, and O_2 if and only if $G \leq \mathrm{Sym}(\Omega)$ is transitive, and $\{\omega,\delta\}^g = \{\omega^g,\delta^g\}$, for all $g \in G$ and $\{\omega,\delta\} \in \Omega^{\{2\}}$.*

Proof. Assume that $G \leq \mathrm{Sym}(\Omega^{\{2\}})$ preserves the canonical orbitals O_0, O_1, and O_2. Since $\mathrm{Sym}(\Omega) \leq \mathrm{Sym}(\Omega^{\{2\}})$ is the automorphism group of the graphs $J(\Omega,2)$ and $K(\Omega,2)$,

[2] A permutation group group G is of affine type if $G = V \rtimes G_0$, where V is isomorphic to a finite vector space, and G_0 which is an irreducible subgroup of $\mathrm{GL}(V)$ is the stabilizer of $0 \in V$

[3] A group G is of almost-simple type if there exists a non-abelian simple group S such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$

Row	Soc(G)	G	Information	$\lvert\Omega\rvert$
1	M_{11}	$G = \mathrm{Soc}(G)$		11
2	$\mathrm{PSL}_2(11)$	$G = \mathrm{Soc}(G)$		11
3	$\mathrm{Alt}(7)$	$G = \mathrm{Soc}(G)$		15
4	HS	$G = \mathrm{Soc}(G)$		176
5	Co_3	$G = \mathrm{Soc}(G)$		276
6	$\mathrm{PSL}_2(8)$	$G = \mathrm{P\Sigma L}_2(8)$		28
7	M_n	$M_n \leq G \leq \mathrm{Aut}(M_n)$	$n \in \{11, 12, 22, 23, 24\}$	n
8	$\mathrm{Ree}(q)$	$\mathrm{Ree}(q) \leq G \leq \mathrm{Aut}(\mathrm{Ree}(q))$	$q = 3^{2k+1}$	$q^3 + 1$
9	$\mathrm{Sz}(q)$	$\mathrm{Sz}(q) \leq G \leq \mathrm{Aut}(\mathrm{Sz}(q))$	$q = 2^{2k+1}$	$q^2 + 1$
10	$\mathrm{PSU}_3(q)$	$\mathrm{PSU}_3(q) \leq G \leq \mathrm{P\Gamma U}_3(q)$	$q \neq 2$	$q^3 + 1$
11	$\mathrm{Sp}_{2n}(2)$	$G = \mathrm{Soc}(G)$	$n \geq 3$	$2^{n-1}(2^n + 1)$
12	$\mathrm{Sp}_{2n}(2)$	$G = \mathrm{Soc}(G)$	$n \geq 3$	$2^{n-1}(2^n - 1)$
13	$\mathrm{PSL}_n(q)$	$\mathrm{PSL}_n(q) \leq G \leq \mathrm{P\Gamma L}_n(q)$	$n \geq 2$	$\frac{q^n-1}{q-1}$
14	$\mathrm{Alt}(n)$	$\mathrm{Alt}(n) \leq G \leq \mathrm{Sym}(n)$	$n \geq 5$	n

TABLE 1. Doubly transitive groups of almost-simple type.

$\mathrm{Sym}(\Omega)$ is a 2-closed group. That is, it is the largest subgroup of $\mathrm{Sym}(\Omega^{\{2\}})$ whose orbitals are the edge sets of $J(\Omega, 2)$ and $K(\Omega, 2)$, which are O_1 and O_2, respectively. We deduce that $G \leq \mathrm{Sym}(\Omega) \leq \mathrm{Sym}(\Omega^{\{2\}})$, and so, for any $g \in G$ and $\{\omega, \delta\} \in \Omega^{\{2\}}$, we have

$$\{\omega, \delta\}^g = \{\omega^g, \delta^g\}.$$

It is clear that G then acts transitively and faithfully on Ω.

Conversely, if $G \leq \mathrm{Sym}(\Omega)$ is transitive and $\{\omega, \delta\}^g = \{\omega^g, \delta^g\}$, for all $g \in G$ and $\{\omega, \delta\} \in \Omega^{\{2\}}$, then $G \leq \mathrm{Sym}(\Omega^{\{2\}})$ preserves the canonical orbitals O_0, O_1, and O_2. □

Remark 2.4. The automorphism group of the strongly-regular graph $NO_5^+(4)$, is the group $O_5(4) \rtimes C_2$. This group is transitive and admits a subgroup that is permutation equivalent to $\mathrm{PSL}_2(16)$ acting on the 2-subsets of $\mathrm{PG}_1(16)$. Consequently, we may view the vertices of $NO_5^+(4)$ as the 2-subsets of $\mathrm{PG}_1(16)$. This subgroup $\mathrm{PSL}_2(16)$ naturally acts as a subgroup of automorphisms of $J(17, 2)$ and $K(17, 2)$. As $O_5(4) \rtimes C_2$ is not contained in $\mathrm{Sym}(17)$, the action of $O_5(4) \rtimes C_2$ on the 2-subsets of $\mathrm{PG}_1(16)$ cannot be via the induced action.

3. THE ACTION OF $\mathrm{PSL}_2(q)$ ON 2-SUBSETS OF $\mathrm{PG}_1(q)$

The action of the group $\mathrm{PSL}_2(q)$ on $\mathrm{PG}_1(q)$ is transitive of degree $q + 1$ and its point stabilizers are conjugate to a subgroup $\mathbb{F}_q \rtimes C_{\frac{q-1}{2}}$ if q is odd, and $\mathbb{F}_q \rtimes C_{q-1}$ if q is even. This group is 2-transitive, and therefore, also 2-homogeneous. Now, consider the induced action of $\mathrm{PSL}_2(q)$ on the 2-subsets of $\mathrm{PG}_1(q)$ and let $\mathbf{u} = \langle [1, 0] \rangle$ and $\mathbf{v} = \langle [0, 1] \rangle$. Let ω be a primitive element of $\mathbb{F}_q$. Define the elements of $\mathrm{PSL}_2(q)$ given by

$$A = \overline{\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}} \text{ and } B = \overline{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}.$$

Note that A has order $\frac{q-1}{2}$ if q is odd, and $q - 1$ if q is even. The element B has order 2, and $BAB^{-1} = B^{-1}$. The stabilizer of $\{\mathbf{u}, \mathbf{v}\}$ for this action of $\mathrm{PSL}_2(q)$ on the 2-subsets of $\mathrm{PG}_1(q)$ is given by the subgroup

$$\langle A, B \rangle = \left\{ A^i B^j : 0 \leq i \leq q - 1, 0 \leq j \leq 1 \right\},$$

	Number	1	$\frac{q-5}{4}$	1
	Size	1	$q(q+1)$	$\frac{q(q+1)}{2}$
Character	Number	$\overline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$	$\overline{\begin{bmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{bmatrix}}$ $1 \le i \le \frac{q-5}{4}$	$\overline{\begin{bmatrix} \sqrt{-1} & 0 \\ 0 & (\sqrt{-1})^{-1} \end{bmatrix}}$
$\rho(\alpha_j)$ $1 \le j \le \frac{q-3}{2}$ even $\alpha_j(\omega) = \exp\left(\frac{2\pi j}{q-1}\mathbf{i}\right)$	$\frac{q-5}{4}$	$(q+1)$	$\alpha_j(\omega^i) + \alpha_j(\omega^{-i})$	$2\alpha_j(\sqrt{-1})$
$\overline{\rho}(1)$	1	q	1	1
$\rho'(1)$	1	1	1	1
$\pi(\chi)$	$\frac{q-1}{4}$	$q-1$	0	0
$\omega_e^{\pm}$	2	$\frac{q+1}{2}$	$(-1)^i$	$(-1)^{\frac{q-1}{4}}$

TABLE 2. Character table of $\mathrm{PSL}_2(q)$, for $q \equiv 1 \pmod 4$.

which is isomorphic to D_{q-1} when q is odd, and $\mathrm{D}_{2(q-1)}$ when q is even. There is a unique conjugacy class of subgroups isomorphic to $\langle A, B\rangle$ in $\mathrm{PSL}_2(q)$ for any $q \ge 3$ (see [7]). Since any transitive permutation group is permutation equivalent to its action on the point stabilizer by right multiplication, we conclude therefore that the group $\mathrm{PSL}_2(q)$ acting on the 2-subsets of $\mathrm{PG}_1(q)$ is permutation equivalent to the action of $\mathrm{PSL}_2(q)$ on right cosets of $\langle A, B\rangle$ by right multiplication.

For $q \ge 13$, the subgroup $\langle A, B\rangle$ is a maximal subgroup of $\mathrm{PSL}_2(q)$ (see [12, Corollary 2.2]). From this, we deduce the following lemma.

Lemma 3.1. *The group* $\mathrm{PSL}_2(q)$ *acts primitively on the* 2*-subsets of* $\mathrm{PG}_1(q)$ *if and only if* $q \ge 13$.

For the rest of this section, we determine the rank of the group $\mathrm{PSL}_2(q)$ acting on 2-subsets of $\mathrm{PG}_1(q)$. When q is even, it was shown in [6] that the rank of $\mathrm{PSL}_2(q)$ acting on 2-subsets of $\mathrm{PG}_1(q)$ is $\frac{q}{2}$. Hence, we may assume that q is odd for the remainder of this section.

Let φ be the permutation character of the action of $\mathrm{PSL}_2(q)$ on 2-subsets of $\mathrm{PG}_1(q)$. That is, for any $x \in \mathrm{PSL}_2(q)$, $\varphi(x)$ is equal to the number of 2-subsets of $\mathrm{PG}_1(q)$ fixed by x. The rank of $\mathrm{PSL}_2(q)$ on 2-subsets of $\mathrm{PG}_1(q)$ is the quantity

$$\langle \varphi, \varphi\rangle = \sum_{\phi} \langle \phi, \varphi\rangle^2,$$

where ϕ runs through all irreducible characters of $\mathrm{PSL}_2(q)$.

In order to determine the decomposition of φ into a sum of irreducible characters, we will need parts of the character table of $\mathrm{PSL}_2(q)$ corresponding to the conjugacy classes of elements of $\mathrm{PSL}_2(q)$ fixing a 2-subset. We note that an element $\overline{A}$ of $\mathrm{PSL}_2(q)$ fixes a 2-subset of $\mathrm{PG}_1(q)$ if and only if its representative A in $\mathrm{SL}_2(q)$ either has two distinct eigenvalues in $\mathbb{F}_q$ or is an involution. For $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$, the relevant parts of the character tables of $\mathrm{PSL}_2(q)$ that correspond to elements fixing a 2-subset are given in Table 2 and Table 3, respectively. See [1] for details on the notations used in these tables. A straightforward computation shows the following lemma.

Lemma 3.2. *The rank of* $\mathrm{PSL}_2(q)$ *on* 2*-subsets of* $\mathrm{PG}_1(q)$ *is equal to* $\frac{3q+7}{4}$ *if* $q \equiv 3 \pmod 4$*, and* $\frac{3q+9}{4}$ *if* $q \equiv 1 \pmod 4$.

	Number	1	$\frac{q-3}{4}$	1
	Size	1	$q(q+1)$	$\frac{q(q-1)}{2}$
Character	Number	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}\omega^i & 0\\ 0 & \omega^{-i}\end{bmatrix}$ $1 \leq i \leq \frac{q-3}{4}$	$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$
$\rho(\alpha_j)$ $1 \leq j \leq \frac{q-3}{2}$ even $\alpha_j(\omega) = \exp\left(\frac{2\pi j}{q-1}\mathbf{i}\right)$	$\frac{q-3}{4}$	$(q+1)$	$\alpha_j(\omega^i) + \alpha_j(\omega^{-i})$	0
$\overline{\rho}(1)$	1	q	1	-1
$\rho'(1)$	1	1	1	1
$\pi(\chi)$	$\frac{q-3}{4}$	$q-1$	0	$-2\chi(\sqrt{-1})$
$\omega_0^{\pm}$	2	$\frac{q-1}{2}$	0	$-(-1)^{\frac{q+1}{4}}$

TABLE 3. Character table $\mathrm{PSL}_2(q)$, for $q \equiv 3 \pmod 4$.

4. PROOF OF THEOREM 1.2

If Γ is a distance-regular graph with a transitive group of automorphisms which is permutation equivalent to $\mathrm{PSL}_2(q)$ acting on 2-subsets of $\mathrm{PG}_1(q)$, then we may assume without loss of generality that its vertex set is the collection of all 2-subsets of $\mathrm{PG}_1(q)$.

Hypothesis 1. *Let p be a prime, $q = p^k$ for some integer $k \geq 1$. Let $\Omega = \mathrm{PG}_1(q)$ and $\Omega^{\{2\}}$ be the collection of all 2-subsets of Ω. In addition, let Γ be a distance-regular graph of order $\binom{q+1}{2}$ defined on $\Omega^{\{2\}}$, and with automorphism group $G = \mathrm{Aut}(\Gamma)$ which admits $\mathrm{PSL}_2(q)$ acting transitively on $\Omega^{\{2\}}$ as a subgroup of automorphisms. Let $r(\Gamma)$ be the rank of the automorphism group of Γ.*

Let Γ be a distance-regular graph as in Hypothesis 1. If $q \leq 11$, then by Lemma 3.1, the action of $\mathrm{PSL}_2(q)$ on $\Omega^{\{2\}}$ is imprimitive. For these graphs, we use `Sagemath` [15] to show that Γ must be one of the graphs in Theorem 1.2. We assume henceforth that $q \geq 13$.

Since $G \leq \mathrm{Sym}(\Omega^{\{2\}})$ is transitive, by Lemma 2.3, either the action of G on $\Omega^{\{2\}}$ is the induced action from a transitive action of G on Ω or G does not preserve the canonical orbitals of Ω defined in (1).

Lemma 4.1. *In addition to Hypothesis 1 and $q \geq 13$, assume that G preserves the canonical orbitals of Ω. Then, the group G is an almost-simple group with socle equal to $\mathrm{PSL}_2(q)$ or $\mathrm{Alt}(q+1)$.*

Proof. From these assumptions and Lemma 2.3, we know that $G \leq \mathrm{Sym}(\Omega) \leq \mathrm{Sym}(\Omega^{\{2\}})$ is transitive. By primitivity of $\mathrm{PSL}_2(q)$ on $\Omega^{\{2\}}$, the group $G \leq \mathrm{Sym}(\Omega^{\{2\}})$ is also primitive. It is clear that $G \leq \mathrm{Sym}(\Omega^{\{2\}})$ cannot be of affine type, and cannot also be as in (3), in Theorem 2.1. Hence, G is a primitive almost simple group contained in the almost simple group $\mathrm{Sym}(\Omega)$. The possibilities for such groups G are given in [13, Table III]. Using Table III in [13] and Table 1, we conclude that the only cases that can occur are Rows 4-5,7,9, Row 13 (for $n = 2$), and Row 14 in Table 1. We immediately rule out the groups in Rows 4-5 since $|\Omega| - 1$ is not a prime power. Let us examine the remaining cases.

- For Row 7 of Table 1, the only values of n for which n is a prime power plus 1 are $n \in \{12, 24\}$. We note that $\mathrm{PSL}_2(11)$ is an imprimitive subgroup of the 2-transitive group M_{12} of degree 12. The group M_{12} acting on the 2-subsets has rank 3, and its

orbital graphs are $K(11,2)$ and $J(11,2)$. However, the automorphism group of these graphs is isomorphic to $\mathrm{Sym}(12)$, which cannot be contained in $\mathrm{Aut}(\mathrm{M}_{12}) = \mathrm{M}_{12}.2$. Similarly, the group M_{24} acting on 2-subsets has rank 3, and its orbital graphs are $K(24,2)$ and $J(24,2)$. Note also that $\mathrm{PSL}_2(23) \leq \mathrm{M}_{24}$. As $\mathrm{Sym}(24)$ is not contained in $\mathrm{Aut}(\mathrm{M}_{24}) = \mathrm{M}_{24}$, this case cannot occur.
- Let $S = \mathrm{Soc}(G)$. For Row 9 of Table 1, $S \trianglelefteq G \leq \mathrm{Aut}(S)$, where $q = 2^{2k+1}$ for some $k \geq 1$, $S = \mathrm{Sz}(q)$, and $\mathrm{Aut}(S) = \mathrm{Sz}(q).C_{2k+1}$. By Hypothesis 1, $\mathrm{PSL}_2(q^2) \leq G$. By noting that $|\,\mathrm{Sz}(q)| = q^2(q^2+1)(q-1)$ it is also immediate that $|\,\mathrm{PSL}_2(q^2)| > |\,\mathrm{Aut}(S)|$, which is impossible.

We conclude that G is an almost simple group with socle equal to $\mathrm{PSL}_2(q)$ or $\mathrm{Alt}(q+1)$. This completes the proof. □

Using Lemma 3.2 and Lemma 4.1, we deduce the following corollary about the diameter of Γ.

Corollary 4.2. If Γ is a distance-regular graph satisfying Hypothesis 1 and $G = \mathrm{Aut}(\Gamma)$ preserves the canonical orbitals, then the diameter of Γ is bounded from above by

$$\begin{cases} 2^{k-1} - 1 & \text{if } p = 2 \\ \frac{3(q+1)}{4} & \text{if } q \equiv 3 \pmod 4, \\ \frac{3q+5}{3} & \text{if } q \equiv 1 \pmod 4. \end{cases}$$

The proof of Theorem 1.2 follows from the next lemma.

Lemma 4.3. *In addition to Hypothesis 1 and $q \geq 13$, assume that $G = \mathrm{Aut}(\Gamma)$ preserves the canonical orbitals of Ω. If Γ is a distance-regular and $r(\Gamma) \leq 11$, then Γ is $J(\Omega,2)$ or $K(\Omega,2)$.*

Proof. Let Γ be a distance-regular graph whose automorphism group has rank $r(\Gamma) \leq 11$. By Lemma 4.1, we have $S = \mathrm{Alt}(q+1)$, in which case $G = \mathrm{Sym}(q+1)$ and Γ is one of $J(\Omega,2)$ or $K(\Omega,2)$, or $S = \mathrm{PSL}_2(q) \trianglelefteq G \leq \mathrm{P\Gamma L}_2(q)$. Hence, it remains to show that there is no such graph Γ whose automorphism group is almost simple with socle equal to $\mathrm{PSL}_2(q)$.

Assume that Γ is such a graph. Since $G \leq \mathrm{Sym}(\Omega^{\{2\}})$ has rank $r(\Gamma)$ and $G \leq \mathrm{P\Gamma L}_2(q)$, the group $\mathrm{P\Gamma L}_2(q) \leq \mathrm{Sym}(\Omega^{\{2\}})$ has rank at most $r(\Gamma) \leq 11$. By (2), we, however, have that

$$r(\mathrm{P\Gamma L}_2(q), \Omega^{\{2\}}) \geq 1 + \frac{\binom{q+1}{2} - 1}{2k(q-1)} = 1 + \frac{p^{2k} + p^k - 2}{4k(p^k - 1)}. \tag{4}$$

For $p \geq 43$ and $k \geq 1$, we have $1 + \frac{p^{2k}+p^k-2}{4k(p^k-1)} \geq 12$, implying that $r(\mathrm{P\Gamma L}_2(q), \Omega^{\{2\}}) \geq 12$. Hence, there exists no distance-regular graphs of this type whose automorphism group has rank at most 11. If $13 \leq q = p^k \leq 41$, we verify with `Sagemath` [15] that there exists no such distance-regular graphs with automorphism group $G \leq \mathrm{P\Gamma L}_2(q)$. □

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